AG 3 - third assignment

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1. Let $X \subseteq \mathbb{P}^n$ be a qp variety, let $U \subseteq X$ be open and let $\{U_i\}_{i \in J}$ be an open cover of U. Prove that a function $h : U \to \mathbb{K}$ is regular on U if and only if the restriction $h_{|U_i}$ is regular for any $i \in J$.

2. Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be qp varieties, and let $\{U_i\}_{i \in J}$ be an open cover of X. Prove that a map $\varphi : X \to Y$ is a morphism if and only if the restriction $\varphi_{|U_i}$ is a morphism for any $i \in J$.

3. Show that every isomorphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is of the form
$$\varphi(x_0 : x_1) = (ax_0 + bx_1 : cx_0 + dx_1),$$
with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{K})$.

4. Let $\varphi : X \to Y$ be a morphism between affine varieties $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, and consider the pull-back $\varphi^\star : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. Prove that $\ker \varphi^\star = I(\varphi(X)) \subseteq \mathcal{O}_Y(Y)$, where $I(\varphi(X))$ denotes the ideal of the image of φ in the ring $\mathcal{O}_Y(Y)$.

5. Prove that the projection
$$\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n, \quad \pi(a_0, \ldots, a_n) = (a_0 : \cdots : a_n)$$
is a morphism of qp varieties.