8 Novembre

Lemma $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Dim. Evidenziamo che $\mathrm{Dom}\left(\frac{\sin x}{x}\right) = \mathbb{R}\setminus\{0\}$

$0 \in (\mathbb{R}\setminus\{0\})'$

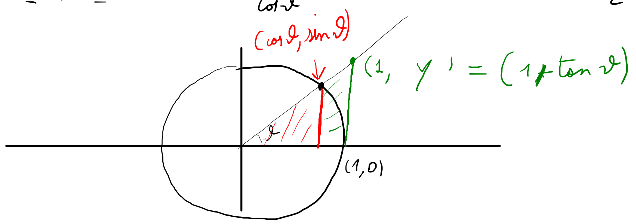
0

Per $0 < x < \frac{\pi}{2}$ abbiamo

$$\sin x \leq x \leq \tan x = \frac{\sin x}{\cos x}$$

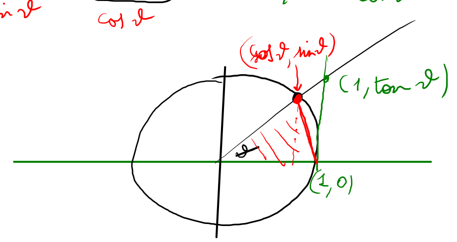
Lo riscrivo nella forma

$$\sin\vartheta \leq \vartheta \leq \tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta} \qquad \text{per } 0 < \vartheta < \frac{\pi}{2}$$



Verifichiamo che $y = \tan\vartheta$. I due triangoli evidenziati sono simili

$$\frac{y}{\sin\vartheta} = \frac{1}{\cos\vartheta} \qquad y = \frac{\sin\vartheta}{\cos\vartheta} = \tan\vartheta$$



$$\frac{\sin\vartheta}{2} \leq \frac{\vartheta}{2} \leq \frac{\tan\vartheta}{2} \Rightarrow \sin\vartheta \leq \vartheta \leq \tan\vartheta \quad \forall\, 0 < \vartheta < \frac{\pi}{2}$$

$$\sin x \leq x \leq \frac{\sin x}{\cos x} \quad \forall\, 0 < x < \frac{\pi}{2}$$ Moltiplico per $\frac{1}{\sin x}$

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \quad \forall\, 0 < x < \frac{\pi}{2}$$

$$\Rightarrow 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \quad \forall\, x \text{ con } 0 < |x| < \frac{\pi}{2}$$

$$1 \xrightarrow{x\to 0} 1 \qquad \frac{1}{\cos x} \xrightarrow{x\to 0} 1 \qquad \Rightarrow \lim_{x\to 0} \frac{x}{\sin x} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{1}{\frac{x}{\sin x}} = \frac{1}{1} = 1$$

Un altro limite notevole è

(1) $$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$$

Si parte da $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$

Osserviamo preliminarmente che

(2) $$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = e,$$

perché se pongo $n=[x]$,

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]}\right)^{[x]} = \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

Data la (2) passiamo alla (1). Ricordiamoci che

$$[x] \le x < [x]+1 \qquad \forall x\in\mathbb{R}.$$

Qui interessa $x >> 1$

$$\frac{1}{[x]} \geq \frac{1}{x} > \frac{1}{[x]+1}$$

$$1+\frac{1}{[x]} \geq 1+\frac{1}{x} > 1+\frac{1}{[x]+1}$$

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} < \left(1+\frac{1}{x}\right)^{[x]} \le \left(1+\frac{1}{x}\right)^{x} \le \left(1+\frac{1}{[x]}\right)^{x} < \left(1+\frac{1}{[x]}\right)^{[x]+1}$$

$$\left(1+\frac{1}{[x]+1}\right)^{[x]} \le \left(1+\frac{1}{x}\right)^{x} < \left(1+\frac{1}{[x]}\right)^{[x]+1} \qquad x>>1$$

$$\lim_{x\to+\infty}\underbrace{\left(1+\frac{1}{[x]}\right)^{[x]}}_{\to e}\underbrace{\left(1+\underbrace{\frac{1}{[x]}}_{\to 0}\right)}_{\to 1} = e.$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]} = \lim_{x\to+\infty}\frac{\left(1+\frac{1}{[x]+1}\right)^{[x]+1}}{\underbrace{1+\frac{1}{[x]+1}}_{\to 1}} =$$

$$= \lim_{x\to+\infty}\left(1+\frac{1}{[x]+1}\right)^{[x]+1} = \lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1} =$$

$$= \lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e$$

Verificare che $\lim_{n\to+\infty} x_{n+1} = \lim_{n\to+\infty} x_n$

$= \lim_{n\to+\infty} x_{n+n_0}$ per ~~un~~ qualsiasi $n_0\in\mathbb{Z}$.

Segue $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$

Abbiamo visto $\lim_{x \to +\infty} \left(1+\frac{1}{x}\right)^x = e$

Vale anche

$$\lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^x = e$$

$$\left(1+\frac{1}{x}\right)^x = \left(1-\frac{1}{y}\right)^{-y} = \left(\frac{y-1}{y}\right)^{-y} = \left(\frac{y-1}{y-1+1}\right)^{-y} =$$

$y=-x$

$$= \left(\frac{1}{1+\frac{1}{y-1}}\right)^{-y} = \frac{1}{\left(1+\frac{1}{y-1}\right)^{-y}}$$

$$= \left(1+\frac{1}{y-1}\right)^{y} = \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)$$

$y=-x$

$$\lim_{x \to -\infty} \left(1+\frac{1}{x}\right)^x = \lim_{y \to +\infty} \left(1+\frac{1}{y-1}\right)^{y-1}\left(1+\frac{1}{y-1}\right)$$

$$= \lim_{y \to +\infty} \left(1+\frac{1}{y-1}\right)^{y-1} = \lim_{y \to +\infty} \left(1+\frac{1}{y}\right)^{y}$$

$$= e$$

Verificare che $\lim_{x \to +\infty} f(x-x_0) = \lim_{x \to +\infty} f(x)$ per qualsiasi x_0

Notazione D'ora innanzi, se $f : X \longrightarrow \mathbb{R}$ $X \subseteq \mathbb{R}$ con $\sup X = +\infty$, $\inf X = -\infty$, se $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x) = L \in \overline{\mathbb{R}}$

scriveremo $\lim_{x \to \infty} f(x) = L$.

$$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^x = e$$

<u>Corollario</u> $\forall x \in \mathbb{R}$ $\lim_{n\to+\infty} \left(1+\frac{x}{n}\right)^n = e^x$

<u>Dim</u> $x \neq 0$

$$\left(1+\frac{x}{n}\right)^n = \left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x$$

$$\lim_{n\to+\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to+\infty}\left(\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right)^x$$

$$y=\frac{n}{x} \qquad = \lim_{y\to\infty}\left(\underbrace{\left(1+\frac{1}{y}\right)^y}_{\to e}\right)^x = e^x$$

stav usando

$$\lim_{z\to z_0} z^\alpha = z_0^\alpha$$

Tasso di interesse

Q_0 soldi iniziali $\qquad t$

tasso $r\%$. Ci sono molti modi di pagare l'interesse.

Una possibilità

$$Q(1) = Q_0 + \frac{r}{100} Q_0 = \left(1+\frac{r}{100}\right) Q_0$$

$$Q(2) = Q(1) + \frac{r}{100} Q(1) = \left(1+\frac{r}{100}\right) Q(1)$$

$$= \left(1+\frac{r}{100}\right)^2 Q_0$$

$$Q_1(t) = \left(1+\frac{r}{100}\right)^t Q_0$$

Un'altra possibilità è la rivalutazione semestrale

$$Q\left(\tfrac{1}{2}\right) = Q_0 + \frac{1}{2}\frac{r}{100} Q_0 = \left(1+\frac{1}{2}\frac{r}{100}\right) Q_0$$

$$Q(1) = Q\left(\tfrac{1}{2}\right) + \frac{1}{2}\frac{r}{100} Q\left(\tfrac{1}{2}\right) = \left(1+\frac{1}{2}\frac{r}{100}\right) Q\left(\tfrac{1}{2}\right)$$

$$= \left(1+\frac{1}{2}\frac{r}{100}\right)^2 Q_0$$

$$Q_2(t) = \left(1+\frac{1}{2}\frac{r}{100}\right)^{2t} Q_0$$

Se invece la rivalutazione avviene n volte all'anno

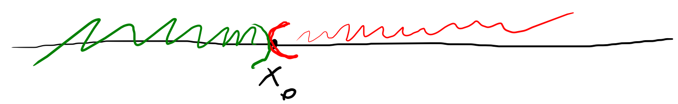
$$Q_n(t) = \left(1+\frac{1}{n}\frac{r}{100}\right)^{nt} Q_0$$

La rivalutazione continua, $n \to +\infty$

$$Q(t) = \lim_{n\to+\infty} \left[\left(1+\frac{1}{n}\frac{r}{100}\right)^n\right]^t Q_0 = e^{\frac{r}{100}t} Q_0$$

Def Sia $f : X \to \mathbb{R}$ $\quad X \subseteq \mathbb{R}$ e sia $x_0 \in X'$. Se $\quad x_0 \in \left(X \cap (x_0, +\infty)\right)'$

e se esiste



$$\lim_{x \to x_0} f|_{X \cap (x_0, +\infty)}(x)$$

$$:= \lim_{x \to x_0^+} f(x) := \lim_{\substack{x \to x_0 \\ x > x_0}} f(x)$$

Analogamente se $\quad x_0 \in \left(X \cap (-\infty, x_0)\right)'$

$$\lim_{x \to x_0^-} f(x) := \lim_{\substack{x \to x_0 \\ x < x_0}} f(x)$$

<u>Esercizio</u> Sia $f : X \to \mathbb{R}$, $\quad x_0 \in \left(X \cap (-\infty, x_0)\right)'$

e $x_0 \in \left(X \cap (x_0, +\infty)\right)'$. Dimostrare che sono equivalenti

1) Esiste $\lim_{x \to x_0} f(x)$

2) Esistono separatamente $\lim_{x \to x_0^+} f(x)$ e $\lim_{x \to x_0^-} f(x)$

e sono uguali.

Inoltre, quando sono vere,

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x).$$