Adding masses

$SU(3)_L \times SU(3)_R$ is broken (explicitly) by quark masses

If $M_q = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$ is the quark mass matrix, this is incorporated into the chiral Lag by $\frac{v}{2} \operatorname{tr}(M_q U + U^\dagger M_q)$

because of sym.

that becomes

$$\mathcal{L}_m = -\frac{v}{f_\pi^2} \operatorname{Tr}\left\{ \pi, \{\pi, M_q\} \right\} =$$

$$= -\frac{v}{f_\pi^2}\Big[4m_u \left(\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0\right)^2 + 4(m_u+m_d)\pi^+\pi^-$$

$$+ 4(m_u+m_s)K^+K^- + 4m_d\left(-\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta^0\right)^2$$

$$+ 4(m_d+m_s)K^0\bar{K}^0 + \frac{8}{3}m_s(\eta^0)^2 \Big]$$

$\pi^0\eta$ mixing disappears when $m_u = m_d$

$$m_{\pi^+}^2 = m_{\pi^0}^2 = \frac{4v}{f_\pi^2}(m_u + m_d)$$

$$m_{K^+}^2 = \frac{4v}{f_\pi^2}(m_u + m_s)$$

$$m_{K^0}^2 = \frac{4v}{f_\pi^2}(m_d + m_s)$$

$$m_{\eta^0}^2 = \frac{4v}{f_\pi^2}\left(\frac{4m_s + m_d + m_u}{3}\right)$$

we cannot directly relate m_π's to m_q's as there is the unknown factor $\frac{v}{f_\pi^2}$

There is also a mixed term $m_{\pi\eta}^2 = \frac{4v}{\sqrt{3}f_\pi^2}(m_u - m_d)$

We can find useful relations however

→ $m_{\pi^0}^2$ scales linearly with quark masses (Gell-Mann / Oakes / Renner)

$\rightarrow \quad \dfrac{m_{K^+}^2 - m_{K^0}^2}{m_\pi^2} = \dfrac{m_u - m_d}{m_u + m_d} \quad \rightsquigarrow$ K^+ and K^0 same masses if $m_u = m_d$

$\rightarrow \quad 3m_\eta^2 + 2m_{\pi^+}^2 - m_{\pi^0}^2 = 2m_{K^+}^2 + 2m_{K^0}^2$

Gell-Mann/Okubo relation (confirmed with good approx by experiments)

$U(1)_A$ problem / η'

$\mathcal{L}_{QCD}$ has another chiral sym, the $U(1)_A$ that rotates

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \mapsto e^{i\gamma_5 \beta} \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

If this was a symmetry and it was spont. broken, it would imply the existence of a pseudo-Goldstone boson with a mass comparable with other π^a.

Now we can do the same steps as before with

$$\pi(x) \mapsto \pi(x) + \frac{f_\pi}{\sqrt{3} f_\zeta} \zeta(x) \mathbb{1}_3 \equiv \pi'$$

Now the mass terms in $\mathcal{L}$ become

$$-\frac{v}{f_\pi^2} \operatorname{Tr}\left\{ \pi', \{\pi', M_q\} \right\} =$$

$$= -\frac{v}{f_\pi^2} \Big[4 m_u \left(\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta^0 + \frac{f_\pi}{\sqrt{3} f_\zeta} \zeta \right)^2 + 4 (m_u + m_d) \pi^+ \pi^-$$

$$+ 4 (m_u + m_s) K^+ \bar{K}^- + 4 m_d \left(-\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta^0 + \frac{f_\pi}{\sqrt{3} f_\zeta} \zeta \right)^2$$

$$+ 4 (m_d + m_s) K^0 \bar{K}^0 + 4 m_s \left(-\sqrt{\frac{2}{3}} \eta^0 + \frac{f_\pi}{\sqrt{3} f_\zeta} \zeta \right)^2 \Big]$$

$\swarrow$

- Charged and strange meson MASSES are the same as before

- Neutral non-strange mesons have now the following mass matrix

$$
\begin{array}{c c}
M_0^2 = \begin{pmatrix}
\frac{m_u + m_d}{2 f_\pi^2} & \frac{m_u - m_d}{2\sqrt{3} f_\pi^3} & \frac{m_u - m_d}{\sqrt{6} f_\pi f_\zeta} \\[2ex]
\frac{m_u - m_d}{2\sqrt{3} f_\pi^2} & \frac{m_u + m_d + 4 m_s}{6 f_\pi^2} & \frac{m_u + m_d - 2 m_s}{3\sqrt{2} f_\pi f_\zeta} \\[2ex]
\frac{m_u - m_d}{\sqrt{6} f_\pi f_\zeta} & \frac{m_u + m_d - 2 m_s}{3\sqrt{2} f_\pi f_\zeta} & \frac{m_u + m_d + m_s}{3 f_\zeta^2}
\end{pmatrix}
&
\begin{array}{l}
\leftarrow \pi^0 \\[4ex]
\leftarrow \eta^0 \\[4ex]
\leftarrow \zeta
\end{array}
\end{array}
$$

In the limit $m_u, m_d \to 0$, M_0^2 has two eigenvector with ZERO eigenvalue

$$
v_a = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_b = \frac{1}{\sqrt{f_\pi^2 + 2 f_\zeta^2}} \begin{pmatrix} 0 \\ f_\pi \\ \sqrt{2} f_\zeta \end{pmatrix}
$$

In these two vector we have at first order in $m_{u,d}$

$$
\begin{array}{c c}
\begin{pmatrix}
\frac{4v(m_u + m_d)}{f_\pi^2} & * \\[3ex]
* & \frac{12 v (m_u + m_d)}{f_\pi^2 + 2 f_\zeta^2}
\end{pmatrix}
&
\begin{array}{l}
\leftarrow u_a \\[5ex]
\leftarrow u_b
\end{array}
\end{array}
$$

where $* = \frac{\sqrt{3} v (m_u - m_d)}{2 f_\pi \sqrt{f_\pi^2 + 2 f_\zeta^2}}$ that is negligible wrt the diagonal terms, that then give the masses with a good approx.

- One sees that $u_a \approx \pi^0$ and $u_b \lessapprox \sqrt{3}\, m_\pi$
- the third eigenvector gives η.

$\Longrightarrow$ A sp. broken $U(1)_A$ sym would require a NEUTRAL PSEUDOSCALAR with MASS $\leq \sqrt{3}\, m_\pi$ in addition to π^0 itself.

$\longrightarrow$ no such particle exist.

THE MESON η' (singlet of $SU(3)_V$)

It has same quark content as η but twice the mass.
This would be the Goldstone boson of the $U(1)_A$ sym.

$U(1)_A$ problem and ANOMALY

Under the axial sym

$$U \mapsto e^{2i\varphi}\, U \qquad (\bullet)$$

We know that the Lagrangian must then effectively be changed by

$$-\frac{2N_f}{32\pi^2}\int d^4x\, \varphi\, \epsilon^{\mu\nu\alpha\beta}\, tr\,(F_{\mu\nu} F_{\alpha\beta})$$

Hence in the effective Lagrangian we should have

$$\underbrace{\frac{i}{2}\left(\log \det U - \log \det U^{\dagger}\right)}_{\delta(\,''\,) = 2i\varphi N_f + 2i\varphi N_f} \frac{1}{32\pi^2}\, \epsilon^{\mu\nu\alpha\beta}\, tr\, F_{\mu\nu} F_{\alpha\beta} =$$

$$= -\frac{\sqrt{2N_f}}{f_\pi}\, \zeta\, \frac{1}{32\pi^2}\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}$$

Let us define the instanton density

$$\rho[A](x) \equiv -\frac{1}{32\pi^2}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}\,(F_{\mu\nu}F_{\alpha\beta}) \quad (A)$$

If the 2-pt function for $\rho[A]$ has the form

$$\langle \rho(x)\,\rho(y)\rangle = m_0^4\,\delta^{(4)}(x-y) + \mathcal{O}(\partial) \quad (*)$$

then η' gets a mass term

$$S_{\zeta_{mass}} = \frac{1}{2}\left(\frac{2N_f m_0^4}{f_\pi^2}\right)\zeta^2 \quad \leftarrow \text{Extra mass due to ANOMALY}$$

(Veneziano-Witten formula)

→ add this term to $\mathrm{tr}\,(M_q U + U^\dagger M_q)$: now one finds only one light neutral meson (π^0) consistently with experiments.

(*) This can be computed by NON-PERTURBATIVE methods like LATTICE QCD.

(A) $\rho[A]$ is a total derivative term, then it should not contribute. However as we will see the config. space of QCD is non-trivial and topologically non-trivial configurations can contribute to it.

There is another interesting pt about this. The anomaly can read

$$\partial_\mu j^\mu_A = 2N_f \frac{1}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}\, F_{\mu\nu} F_{\alpha\beta}$$

As we have said, RHS is a total derivative and it can be written as $-2N_f\, \partial_\mu K^\mu$ with

$$K^\mu = -\frac{1}{8\pi^2} \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}\left(A_\nu \partial_\alpha A_\beta + \frac{2}{3} A_\nu A_\alpha A_\beta \right)$$

$\Rightarrow$ the current $J^\mu_A + 2N_f K^\mu$ is conserved

However K^μ is not gauge inv.

We care on the CHARGE $\int d^3x\, K^0$;

under a gauge transf

$$K^0(A^g) - K^0(A) = -\frac{1}{8\pi^2} \epsilon^{ijk} \partial_i \mathrm{Tr}\,(g^{-1}\partial_j g\, A_k)$$

$$- \frac{1}{24\pi^2} \epsilon^{ijk} \mathrm{Tr}(g^{-1}\partial_i g\; g^{-1}\partial_j g\; g^{-1}\partial_k g)$$

1st term on RHS is a total derivative that does not contribute to the charge.

2nd term matters. However it is non-zero only for non-abelian simmetries (2nd term is non zero for non-trivial g.t.)

Hence we need an anomaly in non-ab. sym to solve the η' problem.

LAGRANGIAN of π^0 DECAY

π^0 can decay into two photons. The Lagrangian that produces a decay like the measured one is

$$\mathcal{L}_{int} = \frac{\alpha}{\pi f_\pi} \bar{E} \cdot \bar{B} \, \pi^0 \quad (*)$$

α is the fine structure const.

$f_\pi \approx 93$ MeV

(measured in leptonic decays of charged pions)

Decay rate

$$\Gamma = \frac{\alpha^2 m_\pi^3}{64 \pi^3 f_\pi^2} \approx 7.63 \text{ eV} \longleftrightarrow$$

Experimental value 7.37 ± 0.15

How can we guess $(*)$ in $\mathcal{L}_{EFT}$ by anomaly considerations?

- We know that under chiral transformations

$$U \mapsto g_L U g_R^\dagger$$

- If we consider an (anomalous) $U(1)_A$ axial tranformation of the form $e^{i\gamma_5 \varphi}$, we obtain

$$\pi^0 \longmapsto \pi^0 + 2 f_\pi \varphi \qquad (*)$$

- How does the action changes under the anomalous transformation? Consider UV theory with only u and d quarks:

then we have

photon

charges under gauged $U(1)_V$ (•)

colours

$$\delta S = \frac{e^2}{8\pi^2}\int d^4x \; \varphi \, F_{\mu\nu}\tilde{F}^{\mu\nu}\left(\left(\frac{2}{3}\right)^2 - \left(-\frac{1}{3}\right)^2\right)\cdot 3$$

(•) from $\mathrm{tr}\, t_V^a t_V^a$

$\to$ We then obtain the effective action (taking into account (∗)):

$$S_{eff} = \frac{\alpha}{4\pi f_\pi}\int d^4x \; \pi^0 F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{\alpha}{\pi f_\pi}\,\vec{E}\cdot\vec{B}\,\pi^0$$