November 9th

<u>Thm</u> E, F B-spaces, $T \in \mathcal{L}(E,F)$

$R(T) = F$ $\quad$ $(R(T) = TE)$.

There $\exists$ $c > 0$ s.t.

$$T(B_E(0,1)) \supseteq D_F(0,c)$$

<u>Corollary</u> Show that this implies that T is open

<u>Pf</u> We first show $\exists$ $c > 0$ s.t.

$$\overline{T(D_E(0,1))} \supseteq D_F(0,2c)$$

We will use the fact F is a Baire space.

$$X_n = n\,\overline{T(D_E(0,1))}$$ they are closed

$$\bigcup_{n=1}^{\infty} X_n = F$$

because $\forall \quad y_0 \in F \quad \exists \; x_0 \in E$

s.t. $\quad Tx_0 = y_0$

$$x_0 \in \underset{E}{B}(0, n) \qquad \text{for } n \geq 1$$

$$= n \, D_E(0,1)$$

$$y_0 = Tx_0 \in T n D_E(0,1) = n \, T D_E(0,1)$$

$$\subseteq \overline{n \, T D_E(0,1)}$$

$$= X_n$$

So indeed $\bigcup_{n\geq 1}^{\infty} X_n = F$

$\star$ For at least one n,

$$\mathring{X}_n \neq \phi$$

$$X_n = \overline{n \, T D_E(0,1)} = n \underbrace{\overline{T D_E(0,1)}}_{X_1}$$

$$\mathring{X}_n \neq \phi \iff \mathring{X}_1 \neq \phi$$

$$\exists \quad y_0 + D_F(0,4c) \subset X_1 = \overline{T D_E(0,1)}$$

$$y_0 \in \overline{T D_E(0,1)}$$

$$-y_0 \in \overline{T D_E(0,1)}$$

$$D_F(0,4c) \subseteq \overline{\overline{T D_E(0,1)} + \overline{T D_E(0,1)}}$$

$$\subseteq \overline{2 T D_E(0,1)}$$

$$* \quad D_F(0,2c) \subseteq \overline{T D_E(0,1)} \qquad 2^{-n-1}$$

$$\Updownarrow$$

$$D_F(0,2^{-n}c) \subseteq \overline{T D_E(0,2^{-n-1})}$$

Let $y \in D_F(0,c) \subset \overline{T D_E(0,\tfrac{1}{2})}$

$\exists \quad z_1 \in D_E(0,\tfrac{1}{2})$ s.t.

$$|y - Tz_1|_F < c\, 2^{-1}$$

$$y - Tz_1 \in D_F(0, 2^{-1}c)$$

Suppose we have found

$z_1, \dots, z_n$ in E

$\|z_1\|_E \le 2^{-1}, \dots, \|z_n\|_E < 2^{-n}$

s.t. $\left\| y - T\sum_{j=1}^{n} z_j \right\|_F \le c\, 2^{-n}$

By induction we can prove $\exists$

z_{n+1} in E $\quad \|z_{n+1}\|_E < 2^{-n-1}$

s.t.
$$\left\| y - T\sum_{j=1}^{n+1} z_j \right\|_F \le c\, 2^{-n-1}$$

$$y - T\sum_{j=1}^{n} z_j \in D_F(0, c\,2^{-n})$$ and by

$$D_F(0, 2^{-n}c) \subseteq \overline{T D_E(0, 2^{-n-1})}$$

$\exists$ $z_{n+1} \in D_E(0, 2^{-n-1})$ st

$$\left\| y - T\sum_{j=1}^{n} z_j - Tz_{n+1} \right\|_F < c\, 2^{-n-1}$$

So we have a sequence $\{z_n\}$ in E

$$\|z_n\|_E \leq 2^{-n}$$

$$\left\|y - T\sum_{j=1}^{n} z_j\right\|_F \leq 2^{-n} c$$

$$x = \sum_{j=1}^{\infty} z_j \qquad \text{the series is convergente}$$

$$\|x\|_E \leq \sum_{j=1}^{\infty} \|z_j\|_E < \sum_{j=1}^{\infty} 2^{-j} = 1$$

So

$$\|y - Tx\|_F = \lim_{n\to+\infty} \left\|y - T\sum_{j=1}^{\infty} z_j\right\|_F = 0$$

$$\boxed{y = Tx}$$

$\forall\ y \in D_F(0,c)\ \exists\ x \in D_E(0,1)$
st. $Tx = y$ $\qquad T D_E(0,1) \supseteq D_F(0,c)$

$\square$

Theo (Closed Graph Theorem)

Def E and F normed space

$E \times F$ becomes a normed space

$$\|(x,y)\|_{E\times F} = \|x\|_E + \|y\|_F$$

Exercise If $T \in \mathcal{L}(E,F)$ then

$$G(T) = \{(x,Tx) : x \in E\}$$

is closed in $E \times F$.

Theorem If E and F are B-spaces and

$T: E \to F$ is linear, then

$$T \in \mathcal{L}(E,F) \underset{\Longleftarrow}{\Longleftrightarrow} G(T) \text{ is closed.}$$

Pf

If $G(T)$ is closed in $E\times F$, since $G(T)$ is also a vector space. Then $G(T)$ also is a B-space

$$\begin{tikzcd} & E \times F \arrow[dl] \arrow[dr] & \\ E & & F \end{tikzcd} \qquad \begin{tikzcd} & G(T) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ E & & F \end{tikzcd}$$

$$\begin{aligned} E &\longrightarrow G(T) \\ x &\longrightarrow (x, Tx) \end{aligned}$$

$$\begin{aligned} G(T) &\longrightarrow E \\ (x,y) &\longrightarrow x \end{aligned}$$

This map is bounded continuous

is bijective and continuous

$$\begin{aligned} E &\longrightarrow G(T) \\ x &\longrightarrow (x, Tx) \end{aligned}$$

is also continuous

$$\begin{tikzcd} & G(T) \arrow[dr, "\pi_2"] & \\ E \arrow[ur, "\mathcal{L}", "x \mapsto (x,Tx)"'] \arrow[rr, "T"'] & & F \end{tikzcd}$$

$T = \pi_2 \circ \mathcal{L}$

π_2 and $\mathcal{L}$ continuous.

$\Rightarrow \quad T \in \mathcal{L}(E, F)$

$$L^2(\mathbb{R}) \qquad \begin{array}{c} L^2(\mathbb{R}) \\ \cup \end{array}$$

$$\frac{d}{dx} \qquad D\left(\frac{d}{dx}\right) \longrightarrow L^2(\mathbb{R})$$

Projections

Def Given E t.v.s, a closed subspace F is / is complementary if $\exists$ a closed subspace G in E s.t. $E = F \oplus G$

$$\left(\begin{array}{ccccc} E & = & F & + & G \\ & & \cup\!\!\!\!\!\!\in & & \cup\!\!\!\!\!\!\in \\ x & = & x_0 & + & x_1 \end{array}, \quad F \cap G = \{0\} \right)$$

Examples 1) If $\dim F < +\infty$

$\Rightarrow F$ is complementary

2) If $\operatorname{codim} F < +\infty$

$\left(\Longleftrightarrow \dim E/_{F} < +\infty \right)$

then F is complementary

3) If $\cancel{H}$ E is Hilbert space

$E = F \oplus F^{\perp}$

4) If $\dim E = +\infty$ E has complete a topology not coming from a structure of Hilbert space, the

$\exists F \subset E$ not complementary.

5) $c_0(\mathbb{N})$ is not complementary in $\ell^{\infty}(\mathbb{N})$

$C^0_0(\mathbb{R}^d) \subsetneq L^{\infty}(\mathbb{R}^d)$ is not complementary in $\uparrow$

Lemma E B-space $\dim F < +\infty$

Then F is is complementary.

Pf $\dim F \leq n < +\infty$

F has a basis $\{f_1, \dots, f_n\}$ and $\forall$

$x \in F$ we have

$$x = x_1 f_1 + \dots + x_n f_n$$

$$(x_1, \dots, x_n) \in \mathbb{R}^n$$

We can define

$$\phi_j : F \longrightarrow \mathbb{R} \qquad \phi_j \in F'$$

$$\phi_j(x) = x_j$$

By Hahn-Banach there are extensions $\phi_j \in E'$

$G = \bigcap_{j=1}^{n} \ker \phi_j$ is closed

We claim that $F + G = E$

$$F \cap G = \{0\}$$

$x \in F \cap G$ $\quad x = x_1 f_1 + \cdots + x_n f_n$

$$x_j = \phi_j(x) = 0 \quad \forall j \Rightarrow x = 0$$

$$E = F + G$$

$x \in E$

$$\phi_j\Big(x - \overbrace{\big(\phi_1(x) f_1 + \cdots + \phi_n(x) f_n\big)}^{\in F}\Big) =$$

$$= \phi_j(x) - \phi_j\big(\phi_1(x) f_1 + \cdots + \phi_n(x) f_n\big)$$

$$= \phi_j(x) - \phi_j(x) = 0$$

$$\Rightarrow \quad x - \big(\phi_1(x) f_1 + \cdots + \phi_n(x) f_n\big) \in G$$

$$\Rightarrow \quad E = F + G$$

Given $\quad E = F \oplus G \simeq F \times G$

$$x = x_1 + x_2$$

$$x_1 = x_1 + 0$$

$$F \times G \longrightarrow F$$

$E \ni x \longrightarrow x_1 \in F \subseteq E$

$$P_1 x \doteq x_1 \qquad P_1 \in \mathcal{L}(E)$$

$$P_1^2 x = P_1(P_1 x) = P_1(x_1) = x_1 = P_1 x$$

P_1 has the property $\quad P_1^2 = P_1$

A projection is $\quad P \in \mathcal{L}(E)$

if $\quad P^2 = P$

If $\quad P \in \mathcal{L}(E)$ is a projection

also $\quad 1 - P$ is a projection

$$(1-P)^2 = (1-P)(1-P) =$$
$$= 1 - 2P + P^2 = 1 - 2P + P =$$
$$= 1 - P$$

Given

P a projection in E, then

$$E = \ker P \oplus \ker(1-P)$$

$$x = Px + (1-P)x$$

$$(1-P)P = P - P^2 = P - P = 0$$

$$P(1-P) = 0$$

$$E = F \oplus G$$

$\boxed{\text{Leroy Projection}}$

$$L^2(\mathbb{T}^d, \mathbb{R}^d) = \underbrace{L^2(\mathbb{T}^d, \mathbb{R}) \times \dots \times L^2(\mathbb{T}^d, \mathbb{R})}_{d \text{ times}}$$

H is the subspace of divergence free vector spaces

$$H = \{ u \in L^2(\mathbb{T}^d, \mathbb{R}^d) : \underline{\nabla \cdot u = 0} \}$$

$$\downarrow$$

$$\hat{u} \in \ell^2(\mathbb{Z}^d, \mathbb{R}^d)$$

$$in \cdot \hat{u}(n) = 0 \qquad \forall n$$

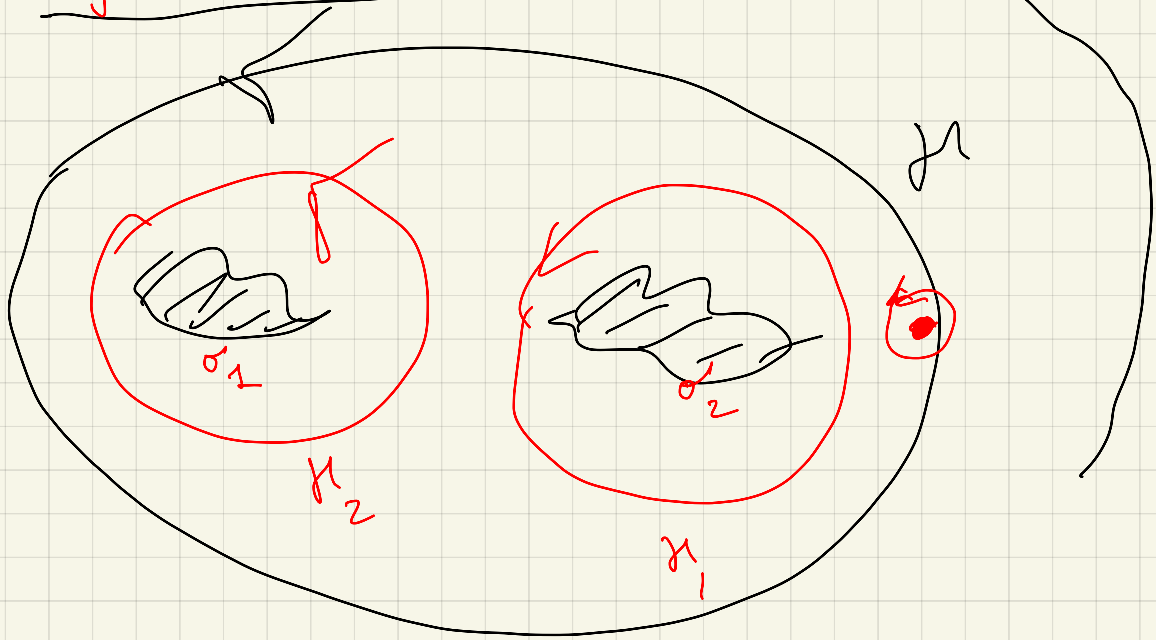
$$L^2 \longrightarrow H$$

$\sigma(\mathbb{P}) \subseteq \{0, 1\}$

$T \in \mathcal{L}(X)$ $\qquad$ X B space in $\mathbb{C}$

$\sigma(T)$

$-\frac{1}{2\pi i} \oint_{\gamma_j} R_T(z)\, dz = P_j$ $\qquad$ $X = \boxed{X_1 \oplus X_2}$



$\sigma(P_1 T) = \sigma_1$ $\qquad$ K

$\sigma(P_2 T) = \sigma_2$