Nov. 11

Def Given a t.v.s E the weak $\sigma(E,E')$ topology is the topology on E which has as a subbasis of seminorms the family

$$\{|f|\}_{f\in E'}$$

Remark Recalling that a t.v.s E is "separable" if it has a countable dense subspace D, it is natural to wonder when E' is separable and $D\subset E'$ is countable and dense, it is natural to ask if the $\sigma(E,E')$ top is equivalent to the one generated by $\{|f|\}_{f\in D}$

If $\dim E = +\infty$, the topology is not the same of the $\sigma(E,E')$

However we will see that in $D_E(0,R)$ the $\sigma(E,E')$ topology coincides with the

Remark A basis of neigh of 0 in $\sigma(E,E')$

$$V = \{ x \in E : |f_j(x)| < \varepsilon, \; f_1, \dots, f_n \in E', \; \varepsilon > 0 \}$$

Exercise The $\sigma(E,E')$ topology is the weakest topology in E for which all the $f \in E'$ are continuous

Exercise $\mathbb{R}$. If E is a t.v.s on $\mathbb{C}$, it is also a v.s on $\mathbb{R}$. And the weak topologies, using either linear functionals in $\mathbb{R}$ ~~or~~

or " " " $\mathbb{C}$,

are the same

Exercise E $\sigma(E,E')$, X a topological space

$F : X \longrightarrow E$,

then F is continuous iff

$f \circ F$ is continuous $\forall f \in E'$.

Lemma E locally convex, the it is Hausdorff for $\sigma(E,E')$.

Pf Let $x_0 \neq x_1$. By Hahn – Banach
$\exists\ f \in E'$ s.t. $f(x_0) < \alpha < f(x_1)$
for some $\alpha \in \mathbb{R}$.

$$f^{-1}((-\infty,\alpha)) \ni x_0$$

$$f^{-1}((\alpha,+\infty)) \ni x_1$$

are open for the $\sigma(E,E')$

Notation ~~A~~ When a sequence $\{x_n\}$ in E ~~with~~ converges ~~to~~ to $x \in E$ in the $\sigma(E,E')$, then we write

$$x_n \xrightarrow{n\to+\infty} x. \qquad \left(x_n \xrightarrow{n\to+\infty} x \right)$$

Lemma E B space and let $\{x_n\}$ be a sequence in E. $x \in E$.

1) $x_n \rightharpoonup x$ for $\sigma(E,E') \iff f(x_n) \longrightarrow f(x)$ $\forall\ f \in E'$

2) $x_n \to x$ strongly $\Rightarrow$ $x_n \rightharpoonup x$

3) $x_n \rightharpoonup x \Rightarrow \{\|x_n\|_E\}$ is bounded and

$$\|x\|_E \leq \liminf_{n\to+\infty} \|x_n\|_E$$

4) $x_n \rightharpoonup x$ and $f_n \to f$ in norm in E'

$\Rightarrow$ $f_n(x_n) \to f(x)$

Pf of 3) For any $f \in E'$

we know that $f(x_n) \to f(x) \Rightarrow$

$$\sup_n \{ |f(x_n)| \} < +\infty \quad \forall f \in E'$$

$x_n \in E''$ with $\sup | \langle x_n, f \rangle_{E'' \times E'} | < +\infty \ \forall f \in E'$.

By Banach Steinhaus $\sup_n \|x_n\|_{E''} = \sup_n \|x_n\|_E < +\infty$

$$\exists \lim_{k \to +\infty} \|x_{n_k}\|_E = \liminf_{n \to +\infty} \|x_n\|_E$$

$$|f(x)| = \lim_{k \to +\infty} |f(x_{n_k})| \le \lim_{k \to +\infty} \|f\|_{E'} \|x_{n_k}\|_E$$

$$= \|f\|_{E'} \lim_{k \to +\infty} \|x_{n_k}\|_E$$

$$\le \cancel{\ldots}$$

$$\underbrace{|f(x)|}_{\|x\|_E} \le \|f\|_{E'} \liminf_{n \to +\infty} \|x_n\|_E \qquad \forall f \in E'$$

$\exists$ $\|f\|_{E'} = 1$

$f(x) = \|x\|_E$

$\Rightarrow$ statement Fatou lemma

Theorem E locally convex t v s and let $C \subseteq E$ be convex. Then are equivalent

1) C closed for the strong topology

2) C " " " $\sigma(E, E')$ top

Pf $2 \Rightarrow 1$. Key is $1 \Rightarrow 2$

Suppose C is strongly closed. Let $x_0 \in E \setminus C$

We can apply Hahn-Banach to C and x_0

$\exists\ f \in E'$ and $\alpha \in \mathbb{R}$ s.t.

$$f(x_0) < \alpha < f(x) \qquad \forall\ x \in C$$

But ~~then~~ $V = \{ y \in E : f(y) < \alpha \}$ is open for the $\sigma(E, E')$ top with $V \cap C = \emptyset$

Then x_0 is not an accumulation point for C in the $\sigma(E, E')$ topology.

Then C is closed for the $\sigma(E, E')$ top.

Remark The above is not true in E' for the $\sigma(E', E)$ topology we will introduce later

$\sigma(E', E'')$. $c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$

<u>Lemma</u> $\dim E = +\infty$. Let U be open for the $\sigma(E, E')$. Then U contains a line.

<u>Pf</u> $0 \in U$. $\exists$ V open $V \subseteq U$ of the form

$$V = \{ x \in E : |f_j(x)| < \varepsilon, \ j = 1, \dots, n \}$$

for some $f_1, \dots, f_n \in E'$ and $\varepsilon > 0$.

Let $F = (f_1, \dots, f_n) : E \longrightarrow \mathbb{R}^n$

codim ker $F \leq n$ ~~and~~ $\dim \ker F = +\infty$

and $\ker F \subseteq V \subseteq U$

<u>Corollary</u> $\dim E = +\infty$, B-space. Then E is not metrizable for $\sigma(E, E')$ top

<u>Pf</u> Suppose $(E, \sigma(E, E'))$ has a metric d.

$D_{\frac{1}{n}} = \{ x : d(x, 0) < \frac{1}{n} \}$

$\exists$ each $D_{\frac{1}{n}}$ is open for $\sigma(E, E')$ contains a line and on this line there exists an $x_n \in D_{\frac{1}{n}}$ s.t. $\|x_n\|_E = n$

Now $d(x_n, 0) < \frac{1}{n} \implies x_n \to 0$ in E

But we have $\lim\limits_{n\to+\infty} \|x_n\|_E = +\infty$

which cannot be true.

<u>Lemma</u> E infinite dim B-space

Let $S := \{x : \|x\|_E = 1\}$. Then

$$\overline{S}\big|_{\sigma(E,E')} \underset{\subseteq}{=} \overline{D_E(0,1)}$$

<u>Pf</u> $\overline{D_E(0,1)} = \{x : \|x\|_E \leq 1\} \supseteq S$

Let $x_0 \in E$, $\|x_0\|_E < 1$ and consider any neigh V of x_0, we know that V contain a line ℓ

$\|\cdot\|_E : \ell \to [0, +\infty)$

and $\exists\ y_0 \in \ell$ $\|y_0\|_E = 1$

$\exists\ y_0 \in V \cap S$

So $\forall$ neigh V of x_0, $V \cap S \neq \phi$

$\implies x_0 \in \overline{S}\big|_{\sigma(E,E')}$

Examples $\quad L^p(\mathbb{R}^d) \quad 1 < p < +\infty$

$\{x_n\}$ a sequence in $\mathbb{R}^d$ with $x_n \xrightarrow{n\to+\infty} \infty$

Then for any $f \in L^p(\mathbb{R}^d)$ we have

$$f(\cdot - x_n) \longrightarrow 0 \quad \text{in } \sigma(L^p, L^{p'})$$

$$p' = \frac{p}{p-1}$$

If for example $\operatorname{supp} f = K$ compact and

$g \in L^{p'} \qquad \operatorname{supp} g = K_1$ compact

$$\langle f(\cdot - x_n), g \rangle_{L^p \times L^{p'}} = \int_{\mathbb{R}^d} f(x - x_n)\, g(x)\, dx$$

$$= \int_{(x_n + K) \cap K_1} f(x - x_n)\, g(x)\, dx = 0$$

Because for $n >> 1 \quad (x_n + K) \cap K_1 = \phi$

~~For g~~ in general $\quad C_c^0(\mathbb{R}^d)$ is dense in $L^{p'}(\mathbb{R}^d)$

$1 < p' < +\infty \qquad g$

$\Rightarrow \forall \varepsilon > 0 \quad \exists\, \tilde{g} \in C_c^0(\mathbb{R}^d)$ s.t.

$$\| g - \tilde{g} \|_{L^{p'}} < \varepsilon$$

$$\langle f(\cdot - x_n), g \rangle = \langle f(\cdot - x_n), \tilde{g} \rangle + \langle f(\cdot - x_n), g - \tilde{g} \rangle$$

$$\downarrow n \to +\infty$$

$$0$$

$$|\langle f(\cdot - x_n), g - \tilde{g} \rangle| \le |f|_{L^p} \, |g - \tilde{g}|_{L^{p'}} \le \varepsilon |f|_{L^p}$$

$$\limsup_{n \to +\infty} |\langle f(\cdot - x_n), g \rangle| \le \varepsilon \, |f|_{L^p} \qquad \forall \varepsilon > 0$$

$$\Longrightarrow \quad \lim_{n \to +\infty} \langle f(\cdot - x_n, g \rangle \rangle = 0$$

$$\forall f \in L^p \qquad 1 < p < \infty$$

$$f(\cdot - x_n) \rightharpoonup 0$$

$$\|f(\cdot - x_n)\|_{L^p} = \|f\|_{L^p}$$

Example $\quad L^p(\mathbb{R}^d) \quad 1 < p < +\infty$

$\lambda > 0 \qquad f_\lambda(x) = \lambda^{\frac{d}{p}} f(\lambda x)$

$$\|f_\lambda\|_{L^p} = \|f\|_{L^p}$$

$$\|f_\lambda\|_{L^p} = \lambda^{\frac{d}{p}} \|f(\lambda \cdot)\|_{L^p} = \cancel{\lambda^{\frac{d}{p}}} \quad \cancel{\lambda^{-\frac{d}{p}}} \|f\|_{L^p}$$

If $\quad \lambda_n \xrightarrow{n \to +\infty} +\infty$

$$f_{\lambda_n} \rightharpoonup 0 \qquad f \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$$

$$g \in C_c^0(\mathbb{R}^d) \text{ è } L^{p'}(\mathbb{R}^d)$$

$$\langle f_{\lambda_n}, g \rangle_{L^p \times L^{p'}} =$$

$$= \int_{\mathbb{R}^d} \lambda_n^{\frac{d}{p}} f(\lambda_n x) \, g(x) \, dx \qquad \lambda_n x = y$$

$$= \int_{\mathbb{R}^d} f(y) \, g\left(\frac{y}{\lambda_n}\right) dy \quad \lambda_n^{\frac{d}{p} - d}$$

$1 < p' < \infty$

$$= \int_{\mathbb{R}^d} f(x) \, \underbrace{g\left(\frac{y}{\lambda_n}\right)}_{\to g(0)} dy \underbrace{\left(\frac{1}{\lambda_n}\right)^{\frac{d}{p'}}}_{\to 0} = \langle f, g_{\frac{1}{\lambda_n}} \rangle$$

$$g_\mu(x) = \mu^{\frac{d}{p'}} g(\mu x)$$

$$\int_{\mathbb{R}^d} f \qquad 0 \qquad = 0$$

$f_{\lambda_n} \longrightarrow 0$ $\qquad \lambda_n \longrightarrow 0$ $\qquad L^p$

$1 < p < +\infty$

$$\langle f_{\lambda_n}, g \rangle = \langle f, \underset{\downarrow \atop 0}{g_{\frac{1}{\lambda_n}}} \rangle \longrightarrow 0$$

$f \in L^1(\mathbb{R}^d)$ $\qquad$ $L^\infty(\mathbb{R}^d)$

$f(-x_n) \not\to 0$

$\langle f(\cdot - x_n), 1 \rangle = \int f(\cdot - x_n) = \int f$

$D_{L^1}(0, 1)$ is not metrizable for $\sigma(L^1, L^\infty)$

$f(-x_n)\,dx \longrightarrow 0$

$C_0^0(\mathbb{R}^d)$ $\qquad$ $\sigma(L^1, C_0^0(\mathbb{R}^d))$