Image Processing for Physicists

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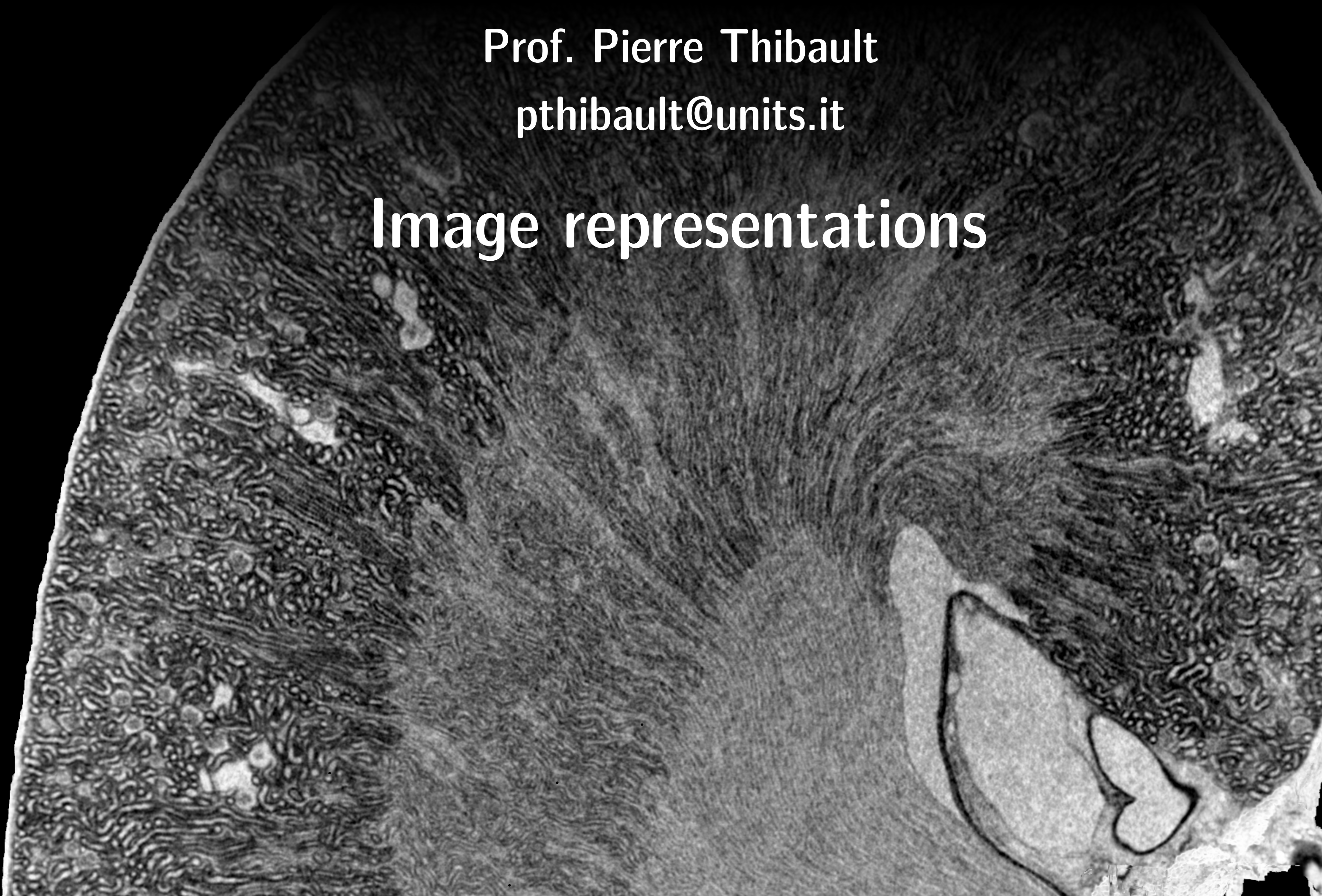
Image representations



Overview

- More on image representations (Fourier-related concepts)
  - DCT Discrete Cosine Transform
  - WFT Windowed Fourier Transform
  - WT Wavelet Transform

Vector spaces

Image representations

$$f(x,y) = \sum c_n B_n(x,y)$$

c_n: coefficients

B_n: basis functions

(most convenient: orthonormal basis)

DFT

$$f(m,n) = \sum_{k,l} F_{kl} \underbrace{e^{2\pi i\left(\frac{mk}{M} + \frac{nl}{N}\right)}}_{B_{kl}(m,n)}$$

DFT basis

$|f\rangle$

$$c_n = \langle B_n | f\rangle$$

$$= \langle B_n | \left(\sum c_m |B_m\rangle\right)$$

$$= \sum_m c_m \langle B_n | B_m\rangle$$

$$= c_n$$

1D: $f_n = \sum_k F_k e^{2\pi i kn/N}$

$$\begin{bmatrix} \\ f \\ \\ \end{bmatrix} = \underbrace{\begin{array}{cc} B_0 & B_1 & & & \\ \hline \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \\ 1 & \vdots & & \ddots & \\ 1 & \vdots & & & \end{bmatrix} \end{array}}_{\mathcal{F}^{-1}} \begin{bmatrix} \\ F \\ \\ \end{bmatrix}$$

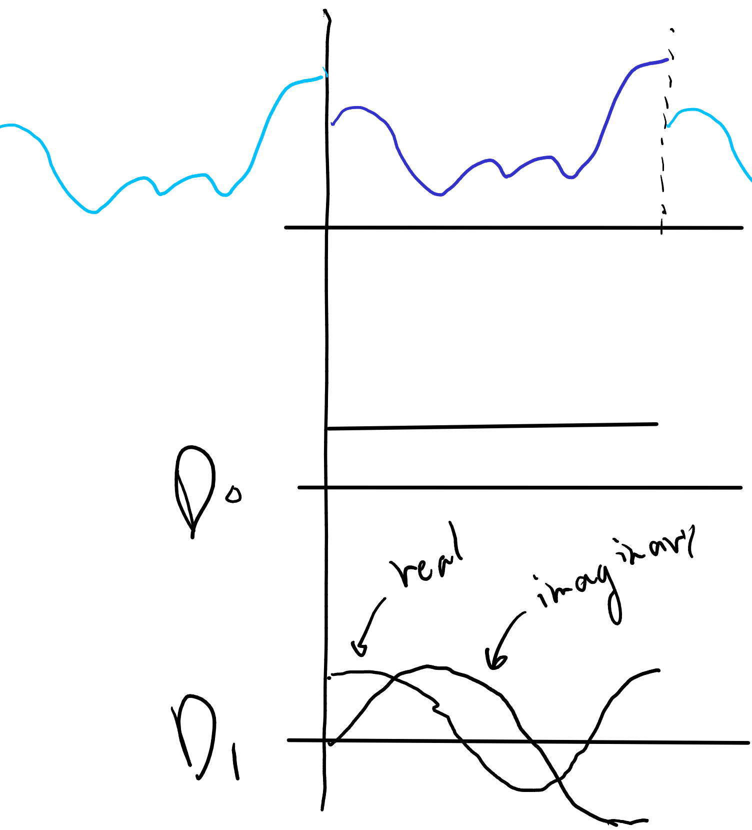
$$\omega = e^{2\pi i/N}$$

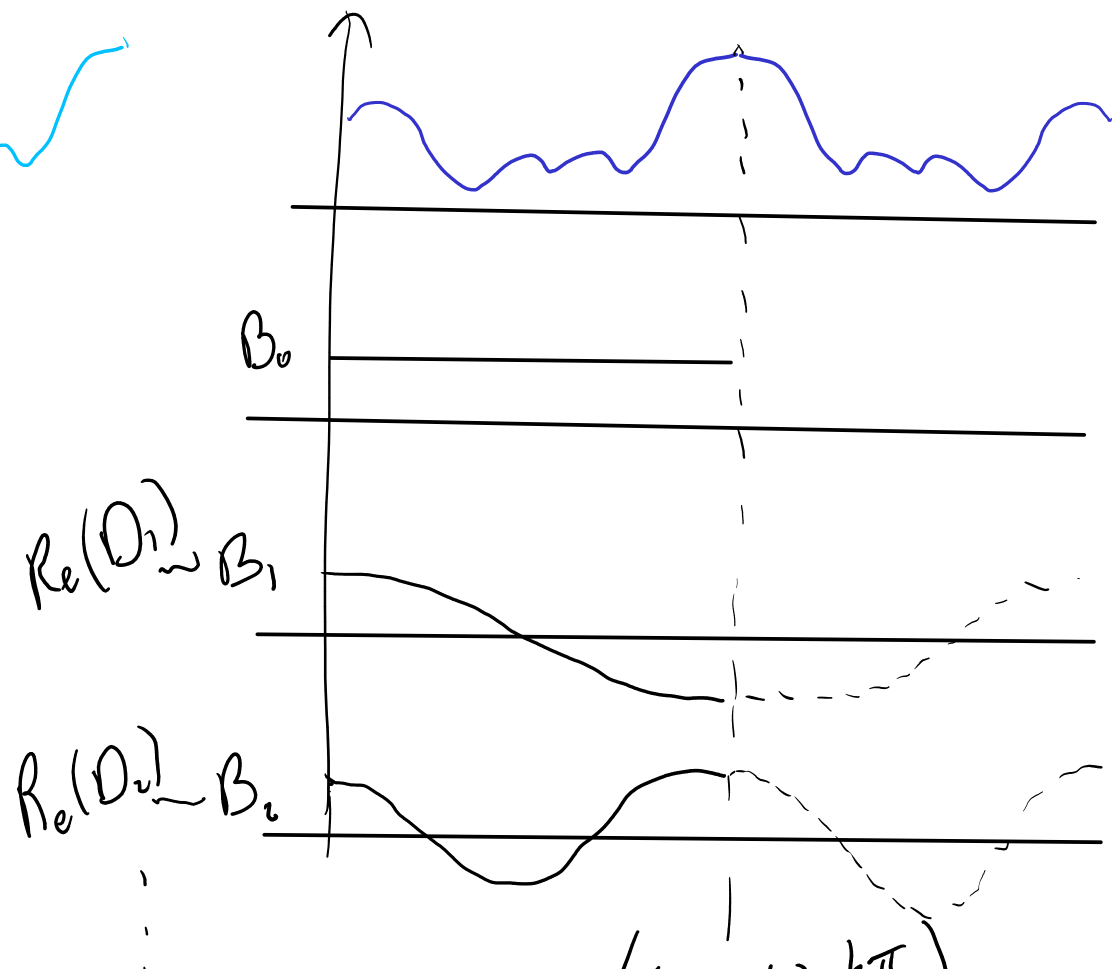
$$\mathcal{F} = \left(\mathcal{F}^{-1}\right)^*$$

DFT is just a change of basis

Discrete Cosine Transform

A variation on the theme of DFT





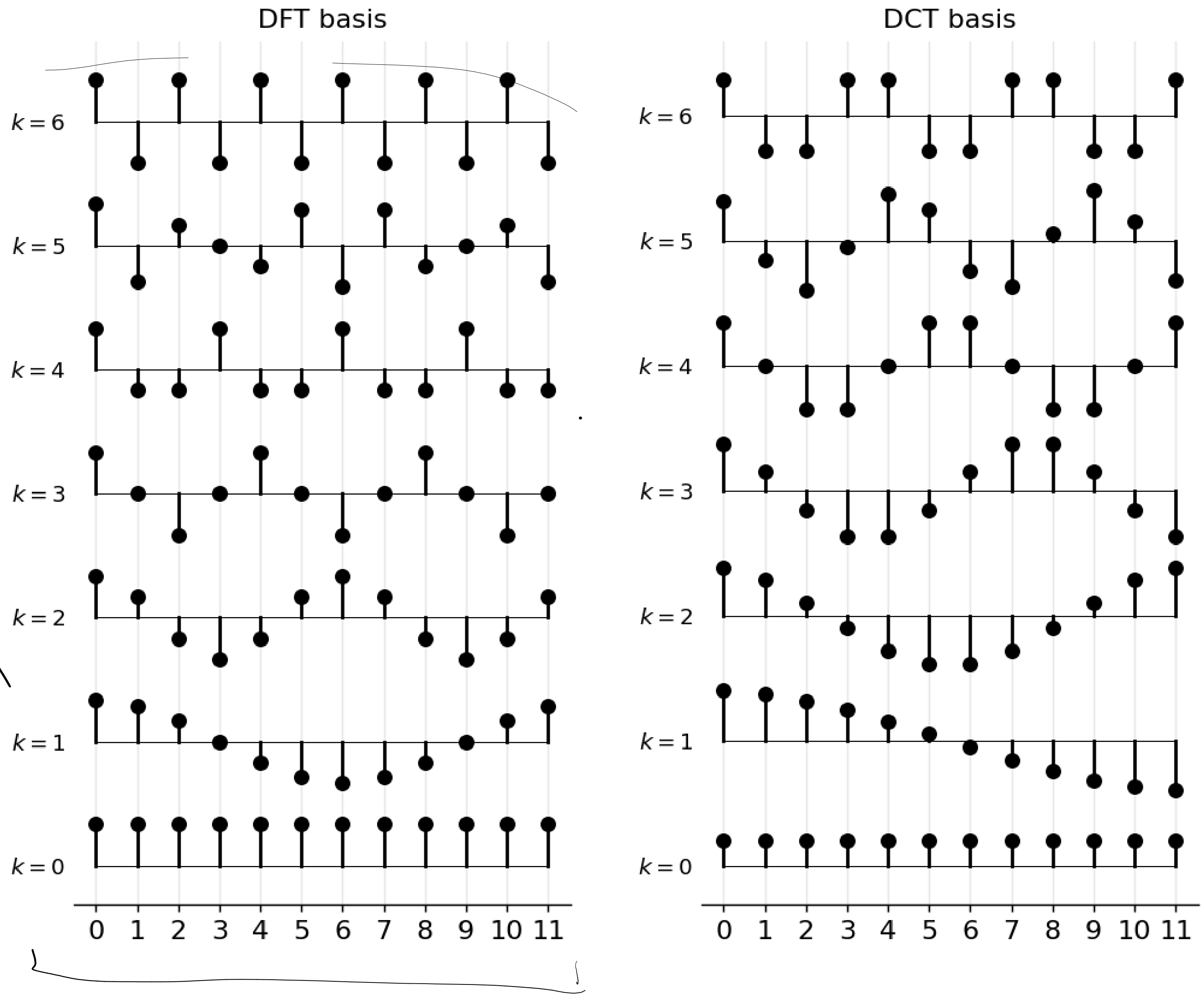
$$D_k(n) = e^{2\pi i k n / N}$$

basis for DFT

$$B_k(n) = \cos\left(\left(n+\tfrac{1}{2}\right)\frac{k\pi}{N}\right)$$

$\left(\times \frac{1}{\sqrt{2}} \text{ for } k=0\right)$

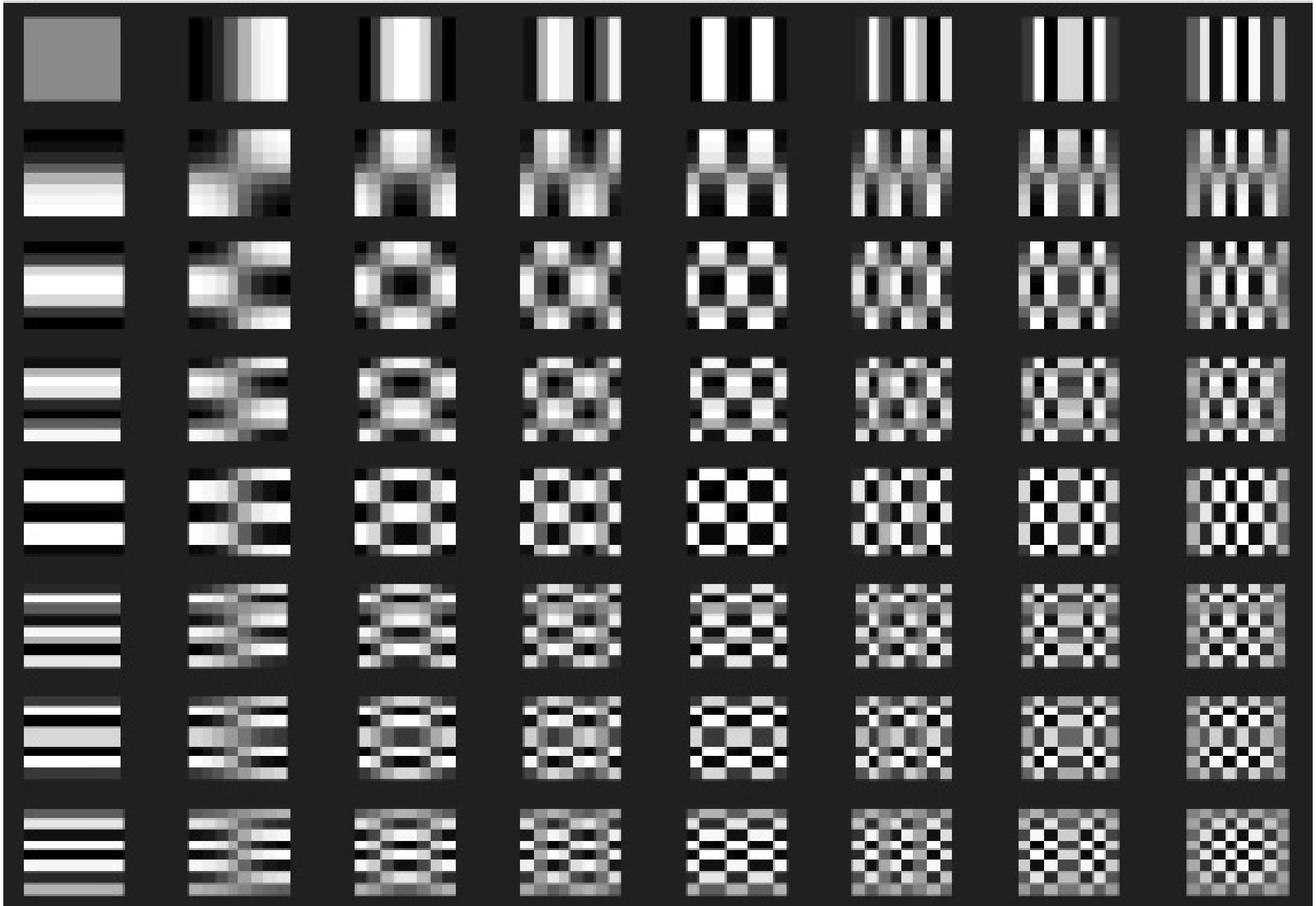
Discrete Cosine Transform



(real) part of DFT

Discrete Cosine Transform

64 DCT basis vectors for 8x8 image



Any 8x8 image is a linear superposition of these 64 basis vectors

Discrete Cosine Transform

Image compression

keeping in average the 8 most significant coefficients out of 64

1:1 bit rate

8:1 bit rate

32:1 bit rate

128:1 bit rate

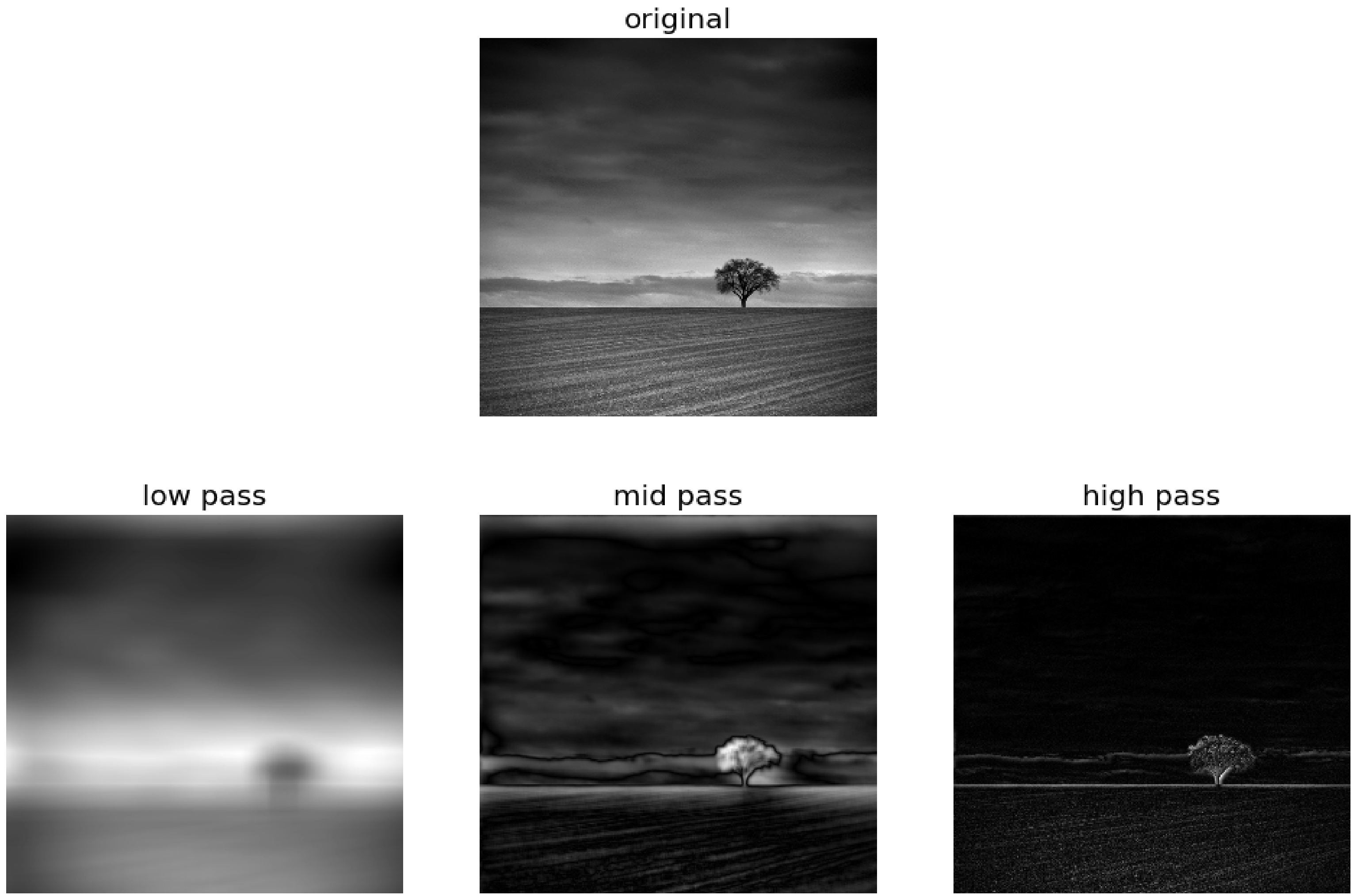
lossy compression

information is discarded

Historical overview

- 1822 Fourier: Fourier transform
- 1946 Gabor: Short-time Fourier transform (STFT)
- 1974 Ahmed, Natarajan & Rao: Discrete Cosine Transform
- 1980s Morlet, Mallat, Daubechies, … : Wavelets

Bandpass filtering

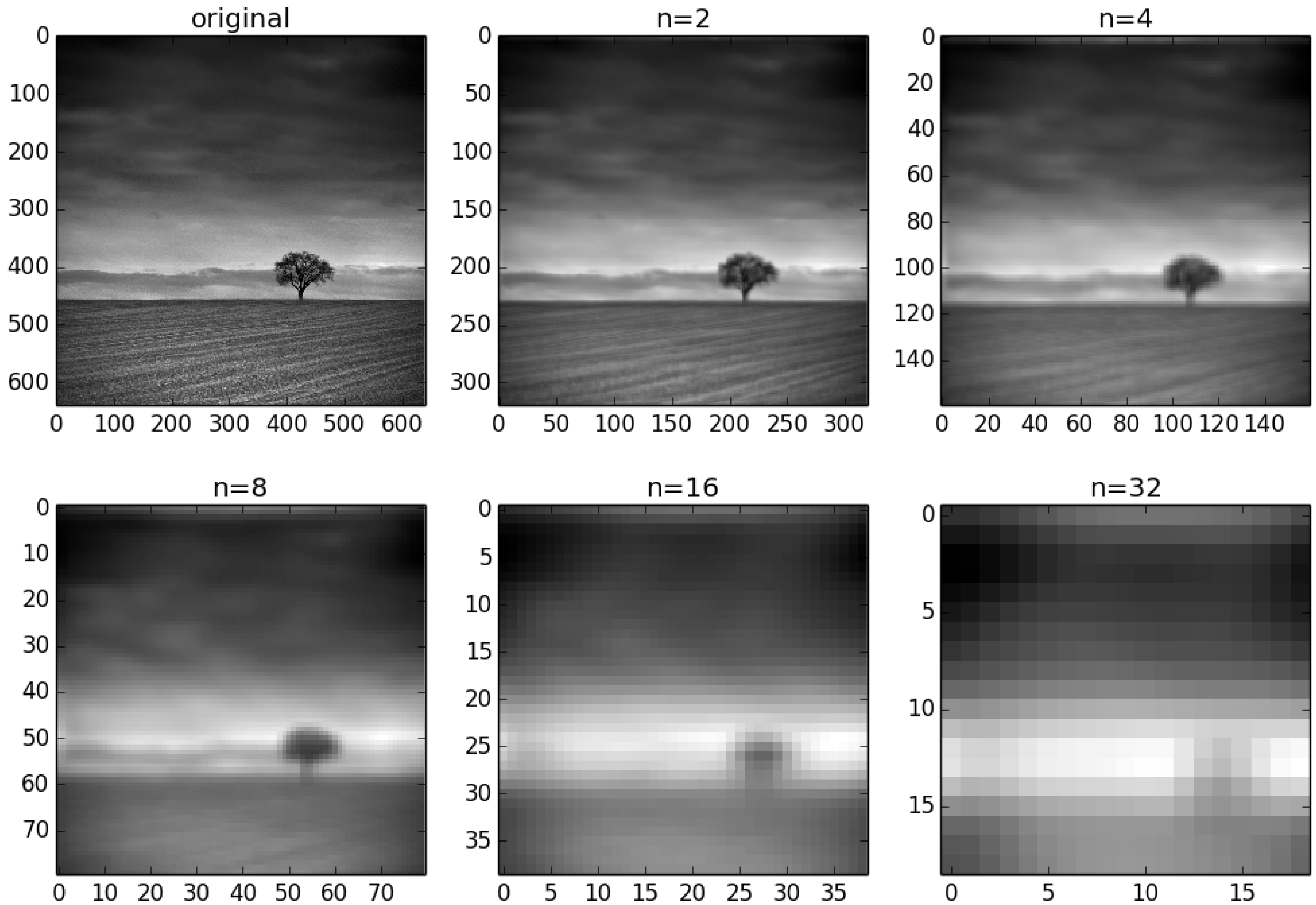


Don't need high spatial resolution

Need high spatial resolution

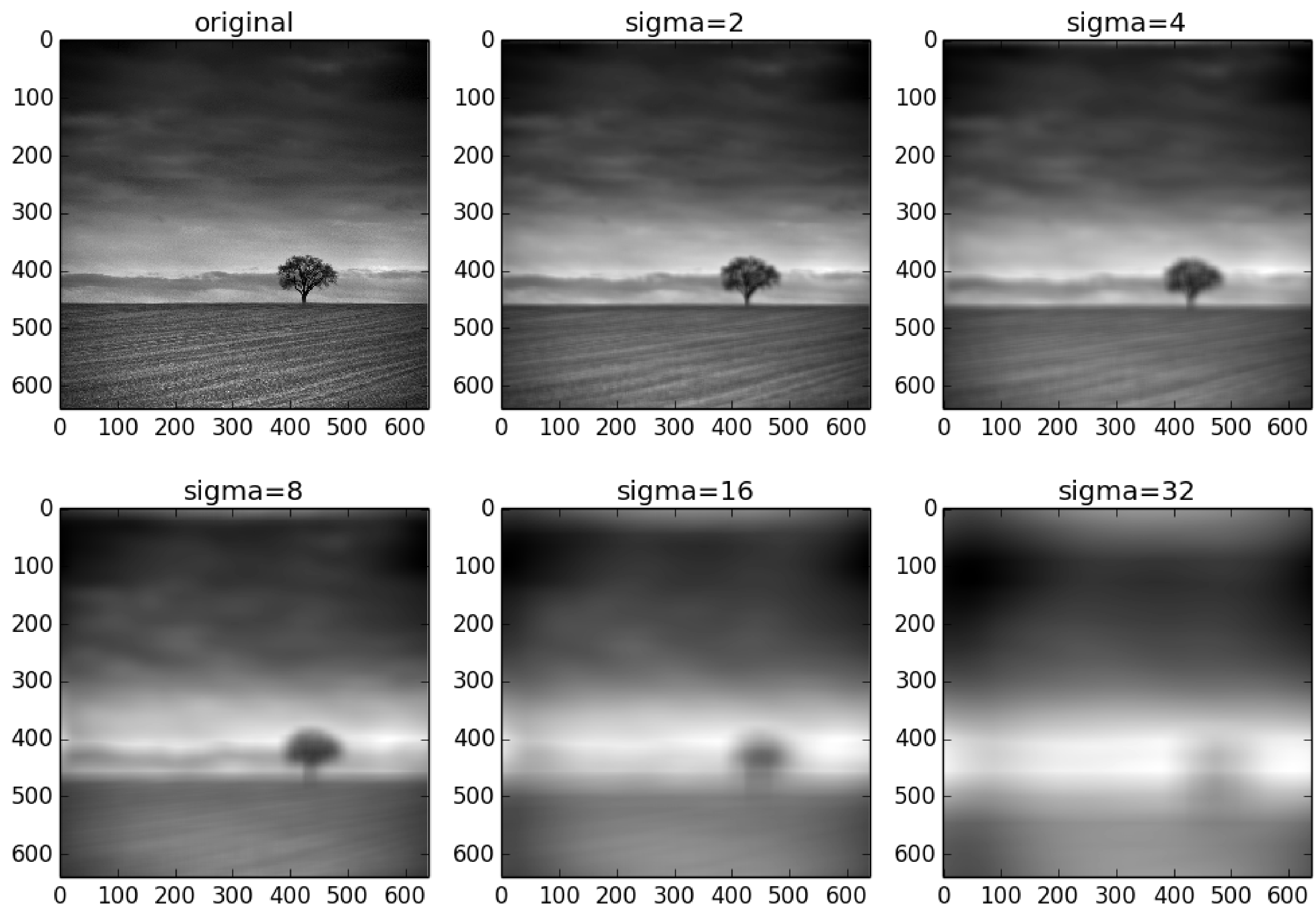
Multiresolution analysis

Subsampling (taking every n^{th} pixel) successively reduces high frequency content



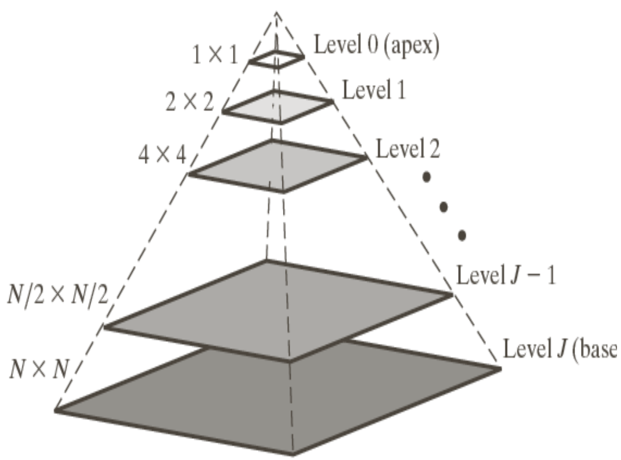
Multiresolution analysis

Multiple filtering with Gaussian filters, sigma determines resolution



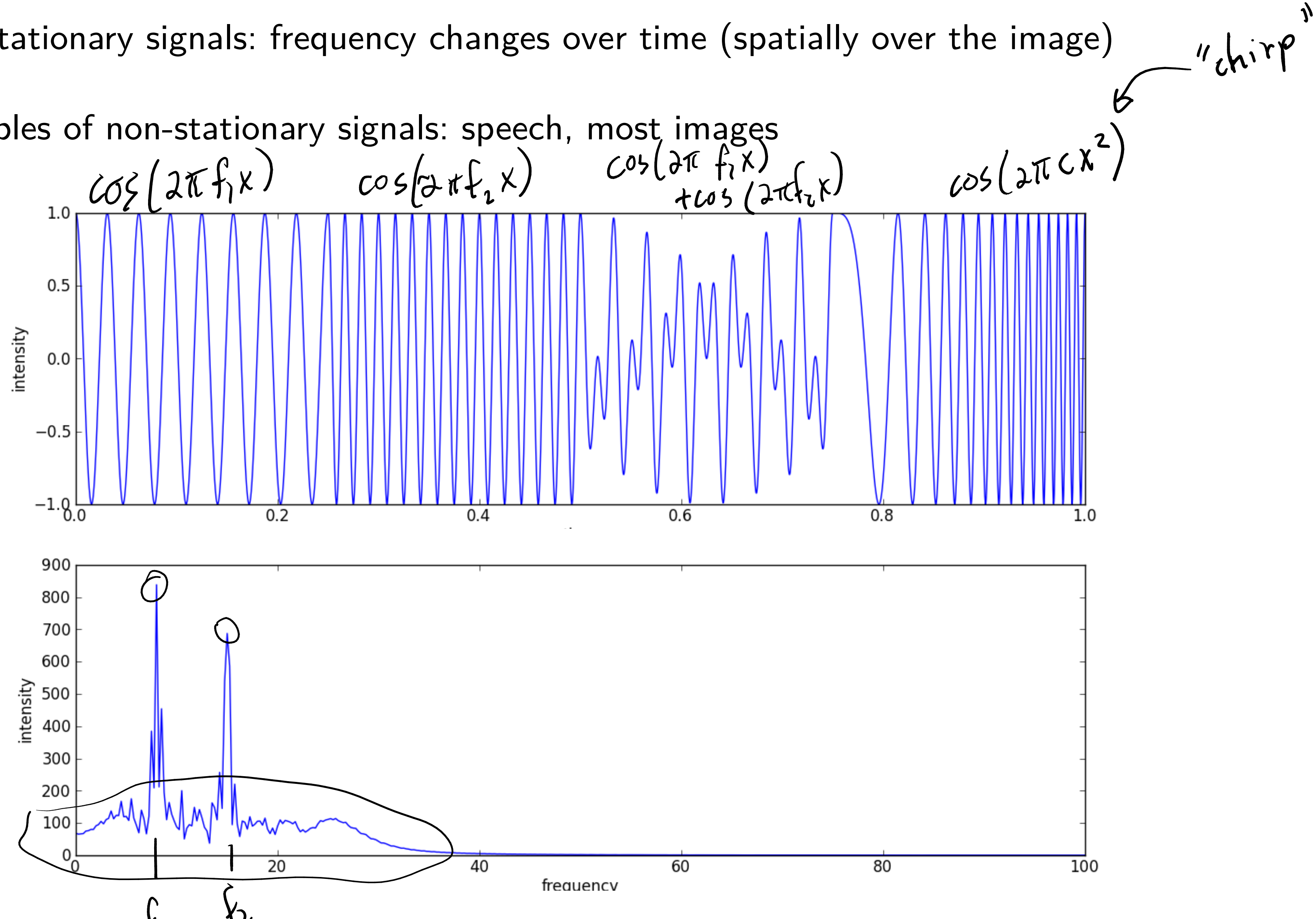
Pyramid representation

Scale-space representation, pyramidal representation



Stationary vs. non-stationary signals

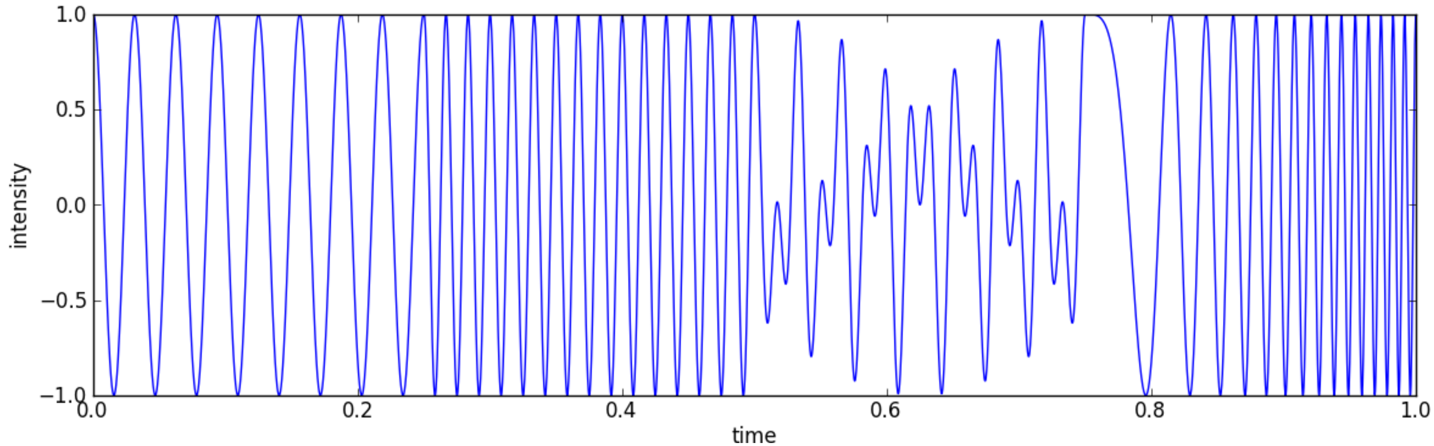
- Stationary signals: frequency doesn't change over time (spatially over the image)
- Non-stationary signals: frequency changes over time (spatially over the image)
- Examples of non-stationary signals: speech, most images



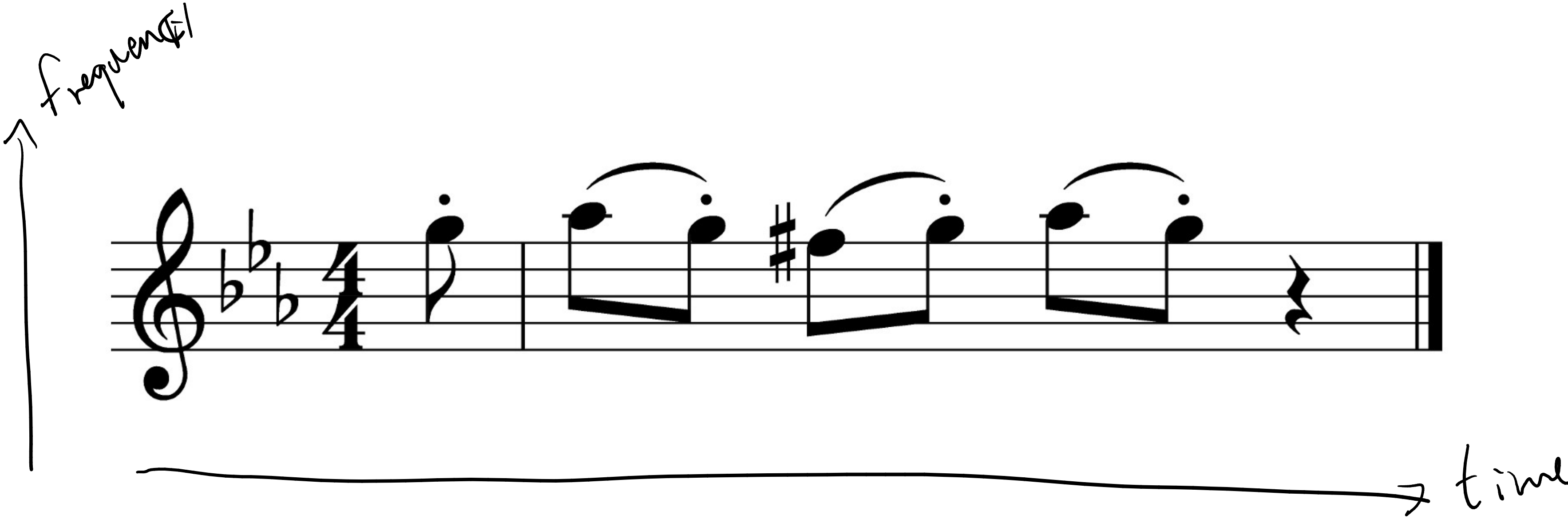
FT insufficient to localize the frequencies in our signal (image)

Windowed Fourier transform

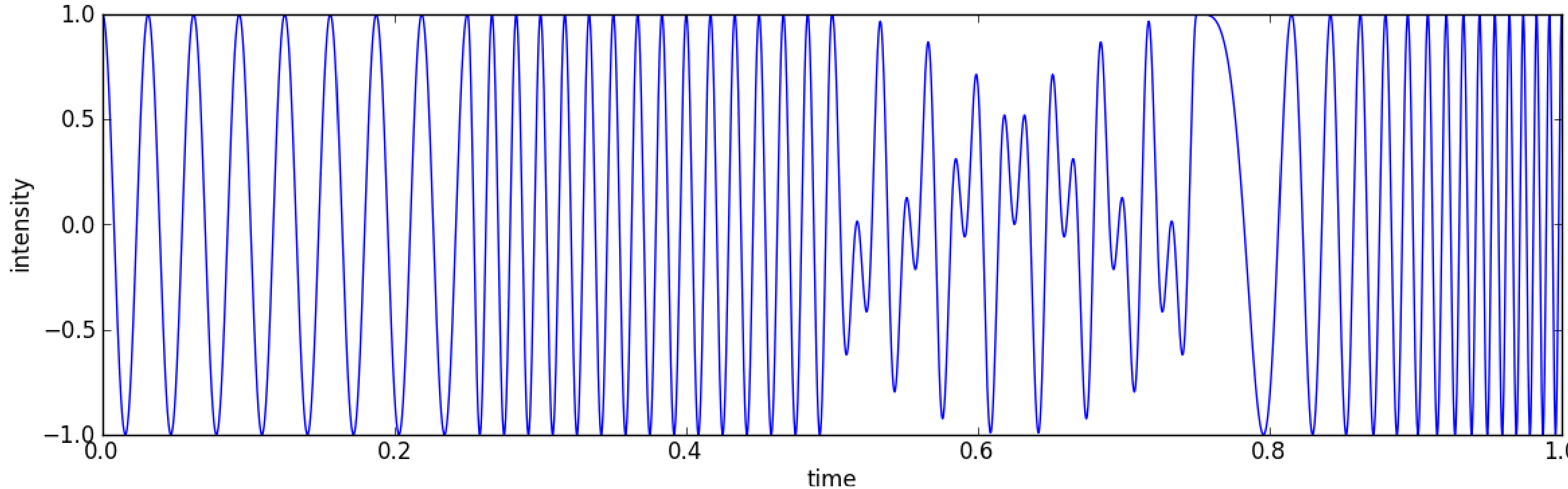
- Windowed Fourier transform is part of the field of "time-frequency analysis"
- Also known as Short-time Fourier Transform (STFT)
- Time-frequency representations are used in many different contexts (Audio, image processing/optics, quantum mechanics)
- Idea: slice up signal into small parts, analyze each separately
  - Multiply with window function w (of width d) at position x0
  - Take Fourier transform of result
  - Slide window to new position
  - repeat

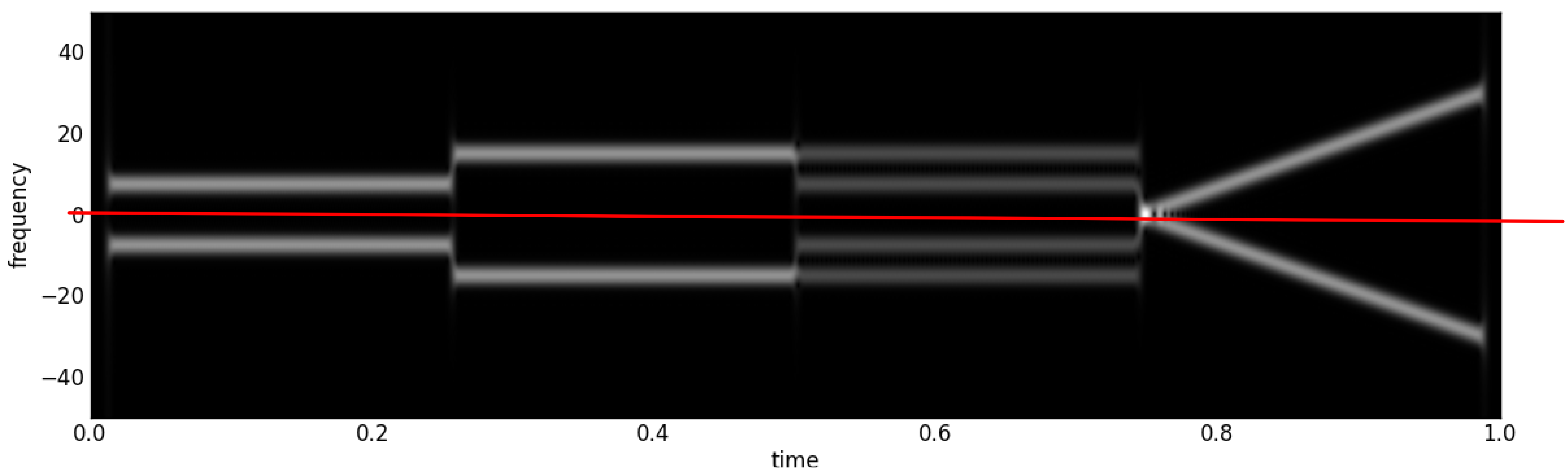


Analogy to audio signals



Spectrogram

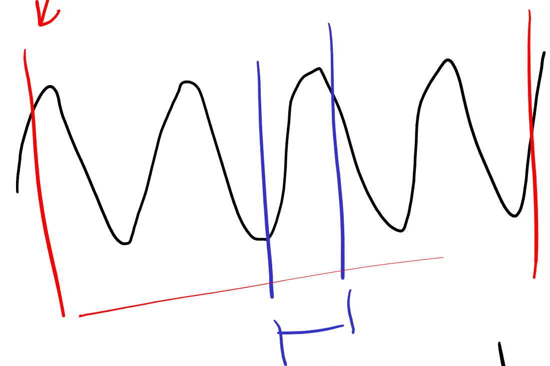




Uncertainty relation

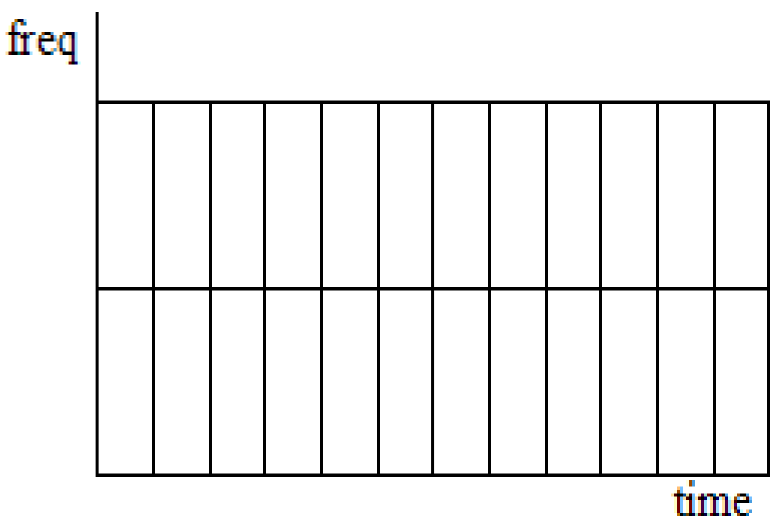
$$\sigma_s \sigma_f \geq \frac{1}{4\pi}$$

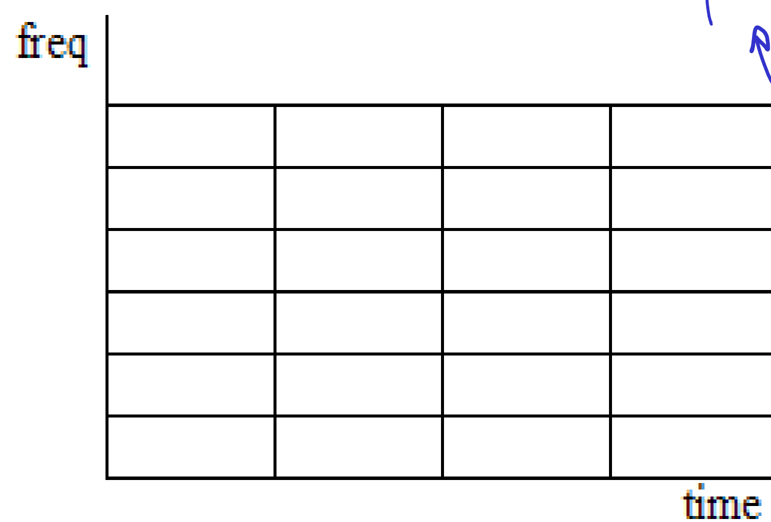
low time res,
high freq. res.



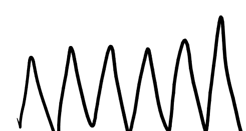
high
time resolution
low freq.
resolution

- Finite area in the time-frequency plane





- This is limitation of WFT and hence development of **wavelets**



higher freq. signal
$\Rightarrow$ can use shorter
time!

Continuous wavelet transform (WT)

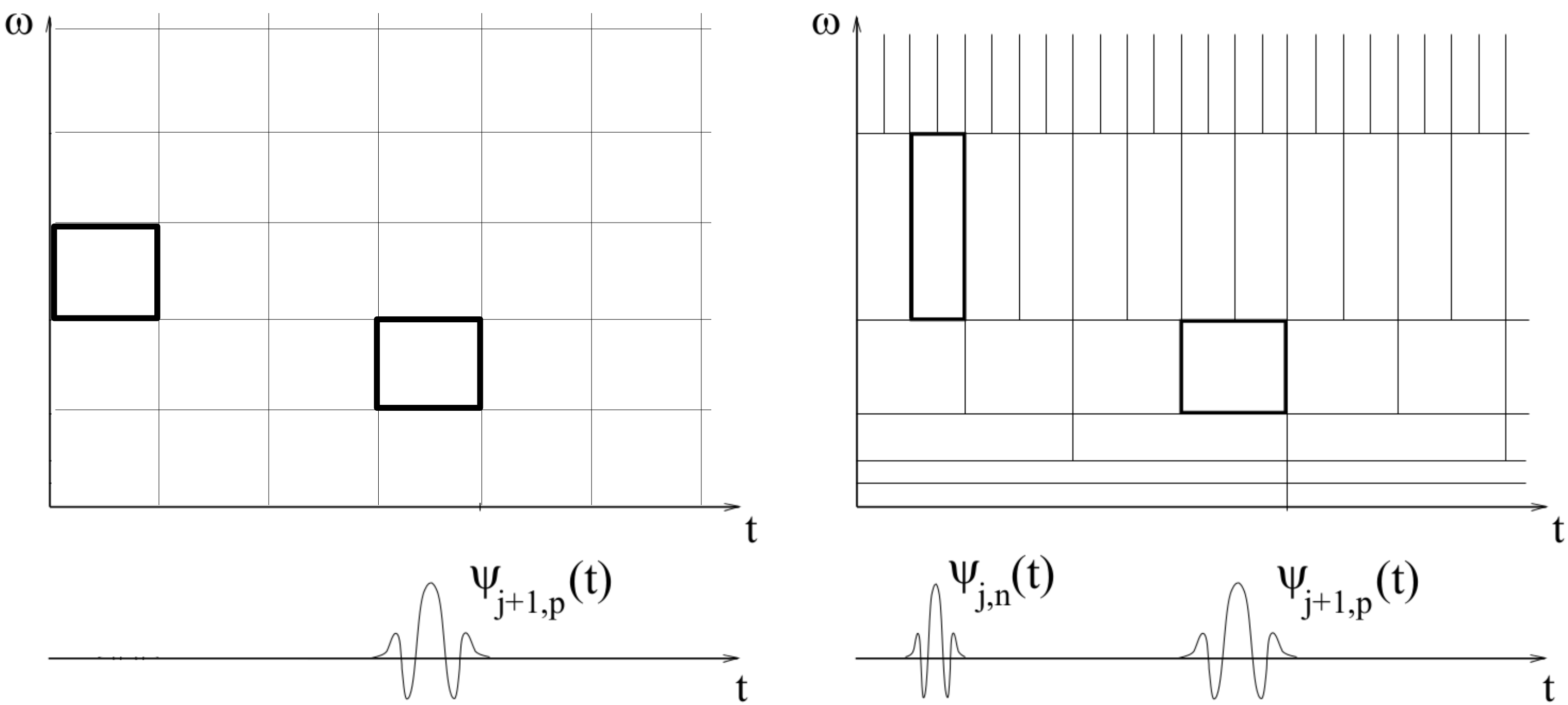
- Parameters: translation and scaling

$$\psi_{s,x_0} = \frac{1}{\sqrt{s}}\, \psi\left(\frac{x - x_0}{s}\right)$$

mother wavelet

$$WT = \int_{-\infty}^{\infty} f(x)\, \psi_{s,x_0}(x)\, dx$$

- Analyze signal at <u>different scales</u> instead of <u>different frequencies</u>

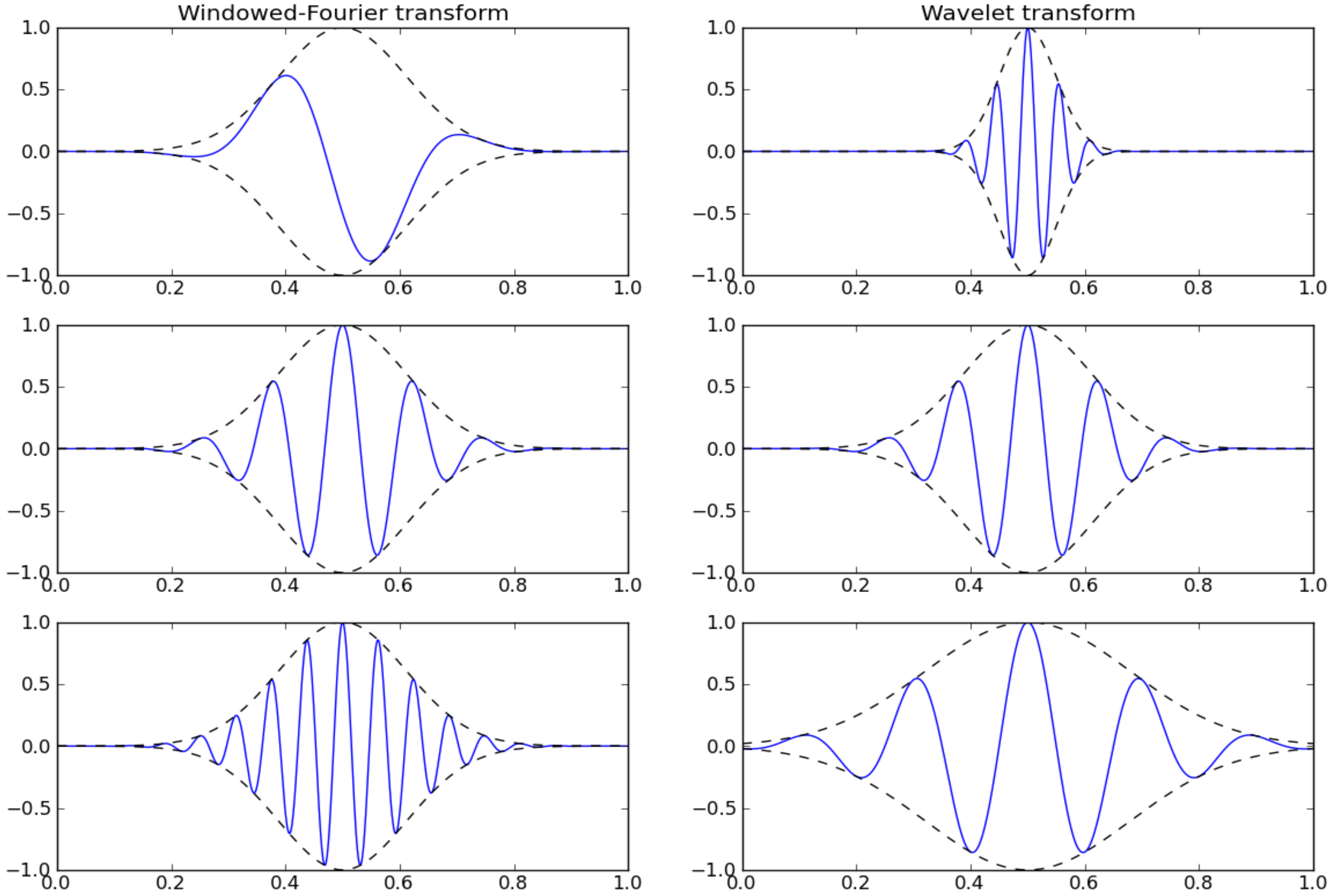


Source: Mallat, "A wavelet tour of signal processing"

WFT vs WT

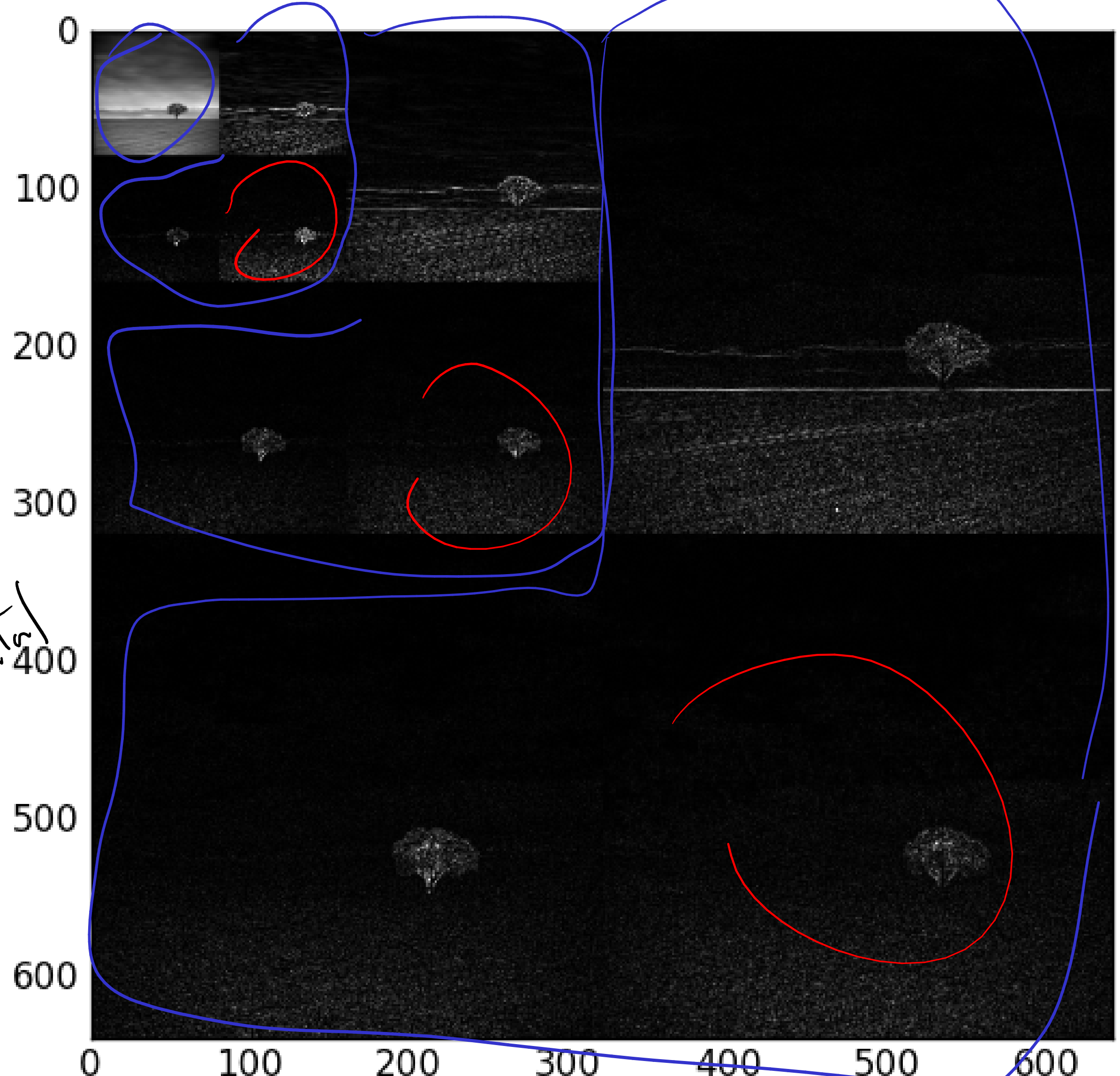
WFT - keep window width constant
- change modulation

Wavelet - keep shape constant
- change scale



Discrete Wavelet decomposition of image

- Perform each DWT, collect and tile all coefficients
- Here: 3 level decomposition



horizontal

vertical

"diagonal"

JPEG 2000 is based on wavelets

Summary

- Images can be represented by different basis functions.
- Fourier basis: localized in frequency, delocalized in real space.
- Windowed Fourier Transform: localized – to some extent – in both spaces
- Wavelet analysis decomposes a signal in position and scale (instead of position and frequency as for WFT).
- Sparse representations are representations in which the image content is represented by a few relevant coefficients, while the other pixels are close to zero
- Sparse representations have advantages for compression, denoising, ...