Image Processing for Physicists

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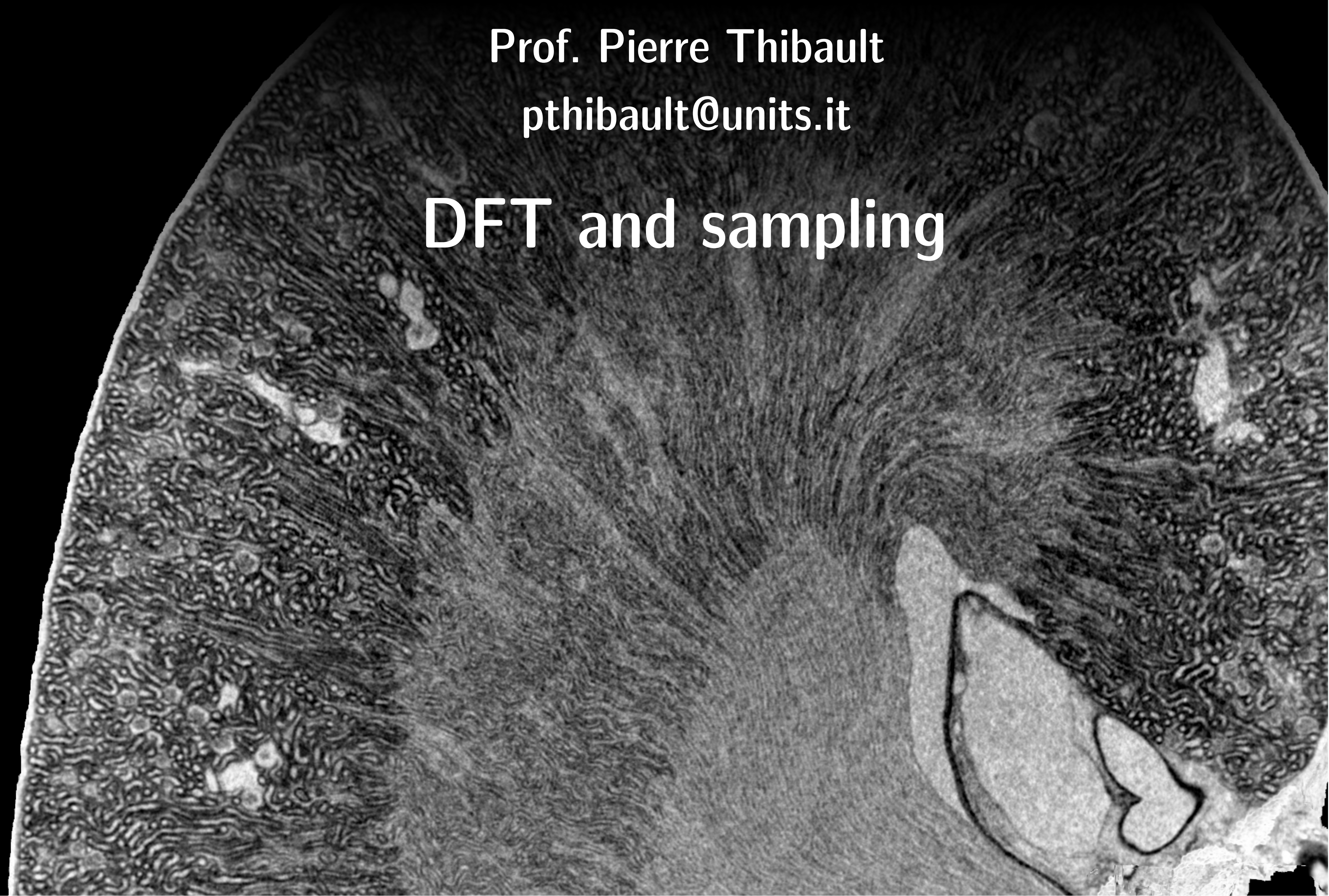
DFT and sampling



Overview

- Discrete Fourier transform
  - Nyquist theorem
  - Undersampling and Aliasing
- Interpolation (resampling)

Reminder: Dirac distribution

- "sifting" property

"selects" value of f at position x_0

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

- normalization

- relation to Fourier transforms

$$\mathcal{F}\{1\} = \int_{-\infty}^{\infty} e^{-2\pi i u x} = \delta(x)$$

Periodic signals

Fourier synthesis equation

$f(x)$: Periodic function with period p

$$f(x) = \mathcal{F}^{-1}\{F\} = \int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, du$$

continous variable $\underline{u}$

but also:

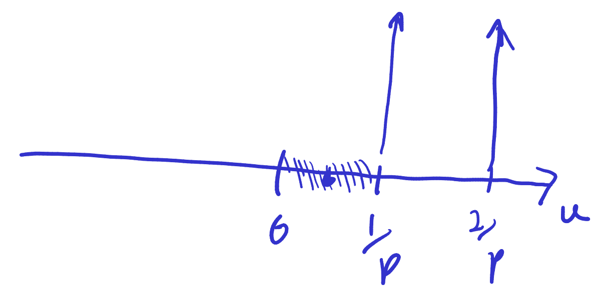
$$f(x) = \sum_{k=-\infty}^{\infty} c_k\, e^{2\pi i k x/p}$$

Fourier series

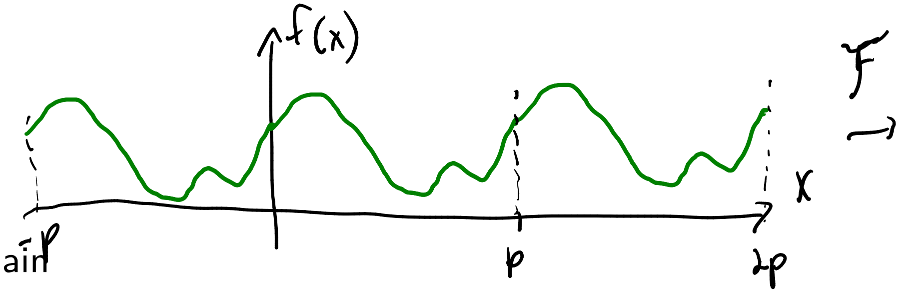
discrete variable

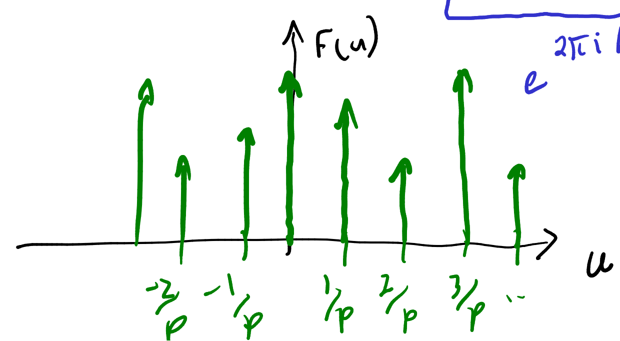
therefore

$$\boxed{F(u) = \sum_{k=-\infty}^{\infty} c_k\, \delta(u - k/p)}$$



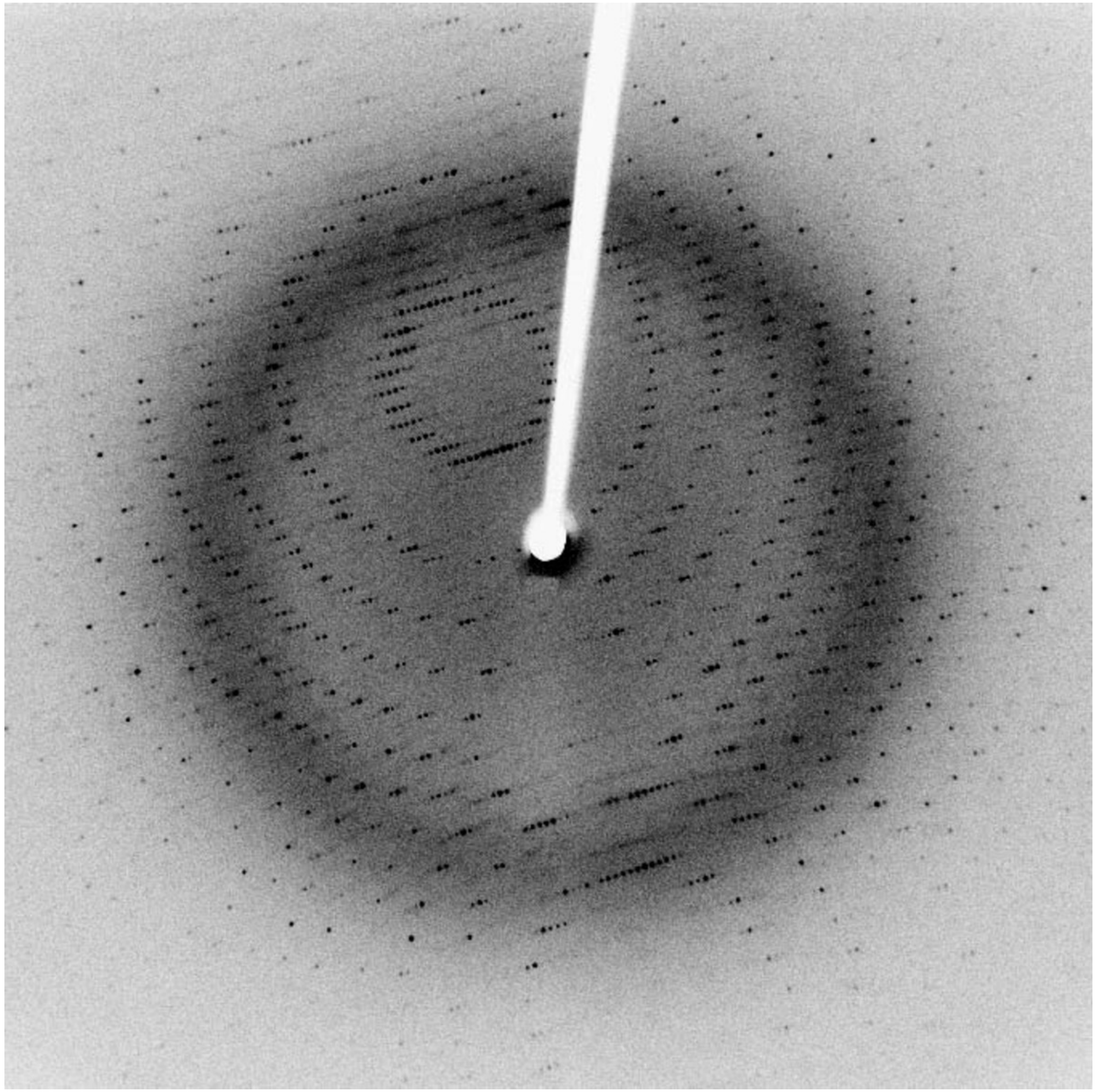
[because. $\displaystyle\int_{-\infty}^{\infty} F(u)\, e^{2\pi i u x}\, du = \int_{-\infty}^{\infty} \sum_k c_k\, \delta(u - k/p)\, e^{2\pi i u x}\, du = \sum_k c_k \underbrace{\int_{-\infty}^{\infty} \delta(u - k/p)\, e^{2\pi i u x}\, du}_{e^{2\pi i k x/p}}$]





Periodic signals

X-ray diffraction by a crystal



"Bragg peaks"

⇕

Dirac deltas caused by periodicity

The Dirac comb

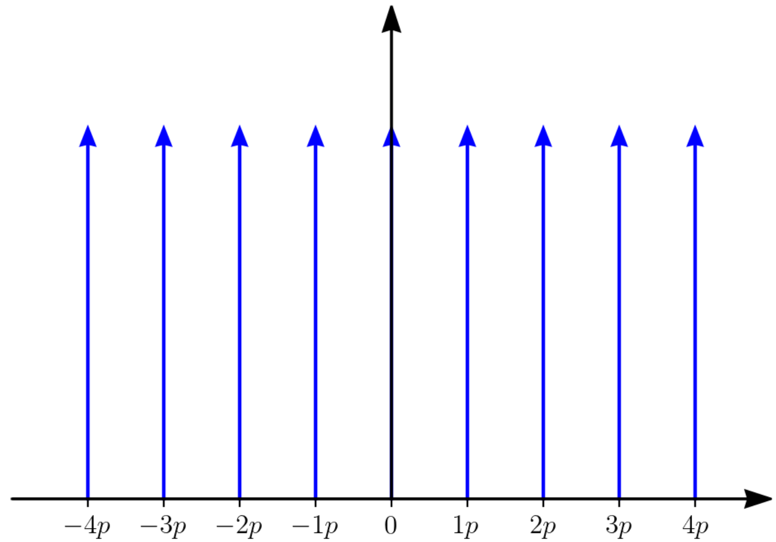
A periodic "function" made of Dirac functions

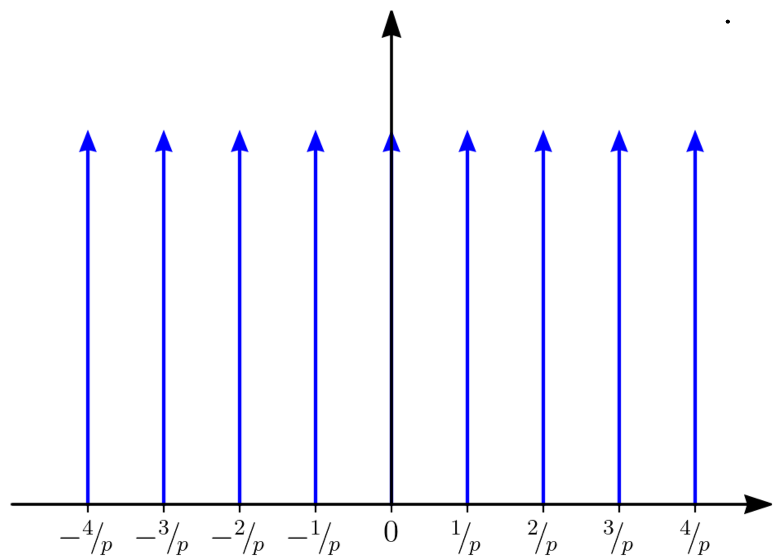
$$\Delta_p(x) = \sum_{n=-\infty}^{\infty} \delta(x - np)$$

$$\Delta_{\frac{1}{p}}(u) = \sum_{k=-\infty}^{\infty} \delta(u - k/p)$$

$$\mathcal{F}\{\Delta_p(x)\} = \frac{1}{p}\Delta_{\frac{1}{p}}(u)$$

Fourier transform of a Dirac comb is a Dirac comb

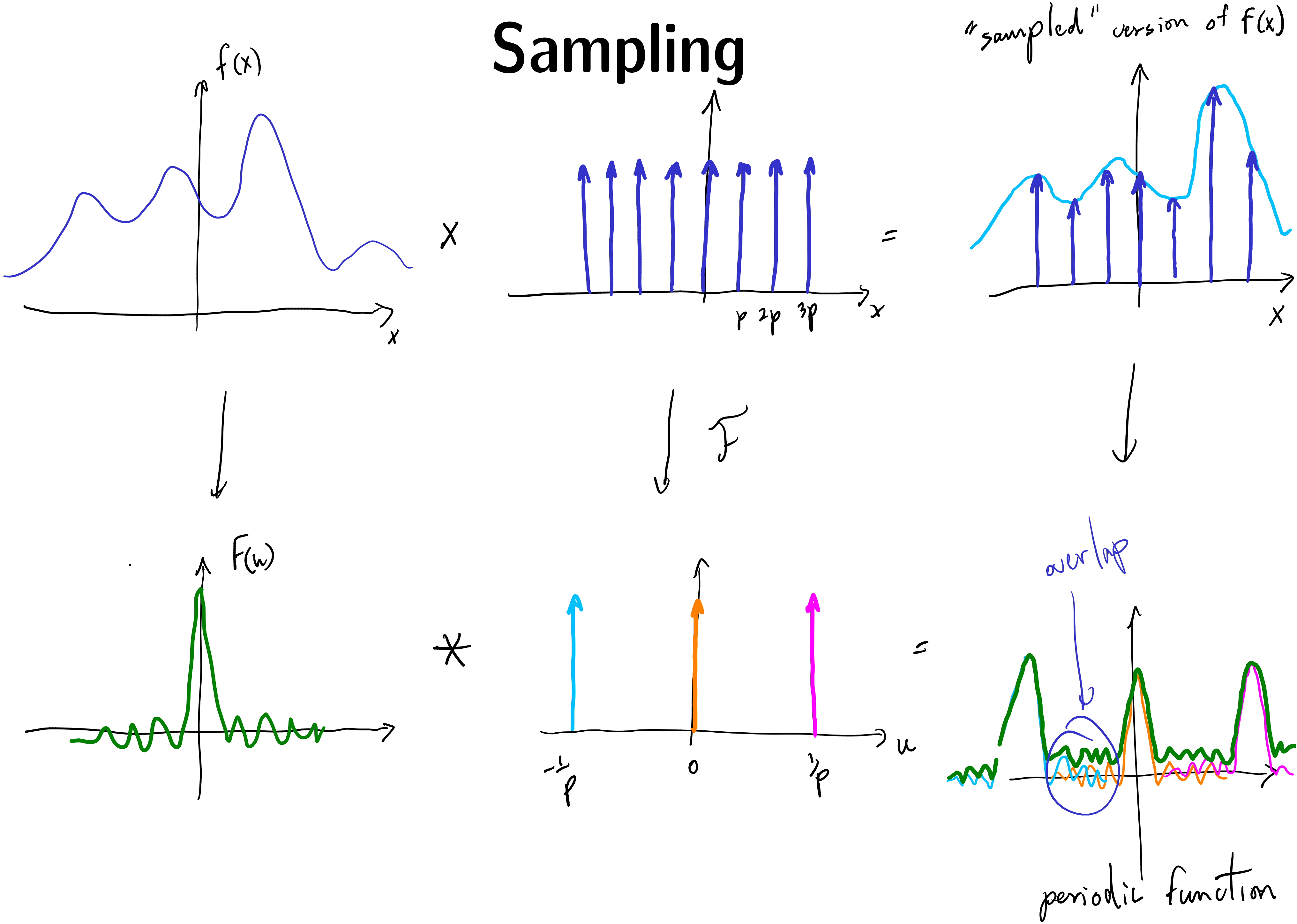




The discrete Fourier transform

- additional ingredients needed:
  - sampling in space
  - finite field of view in space
  - sampling in frequency domain
  - finite frequency band
- discrete approximation of some continuous function

Sampling



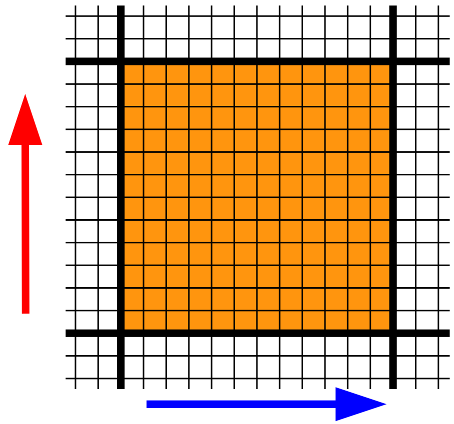
f(x)
x
X
x
p 2p 3p
=
"sampled" version of f(x)
X
F
F(w)
*
-1/p
0
1/p
u
=
overlap
periodic function

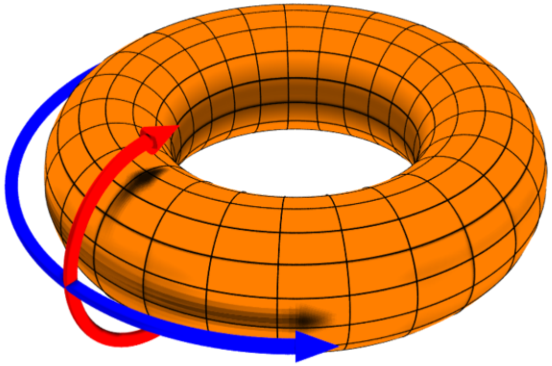
Discrete Fourier Transform

- A **periodic** function has a **discrete** spectrum in the Fourier domain;

- A function with **discrete** values in the spatial domain is **periodic** in the Fourier domain;

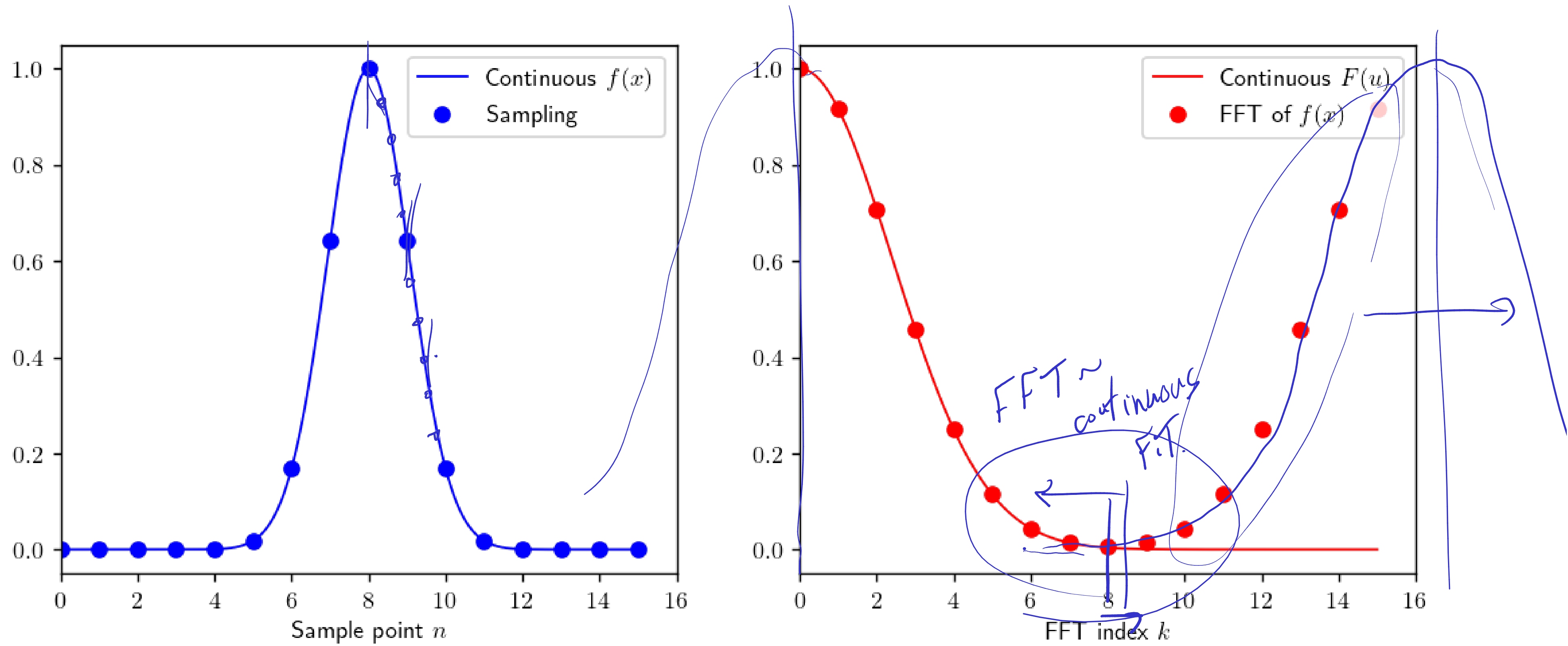
$\Rightarrow$ A periodic and discrete function has a periodic and discrete Fourier transform.





DFT example

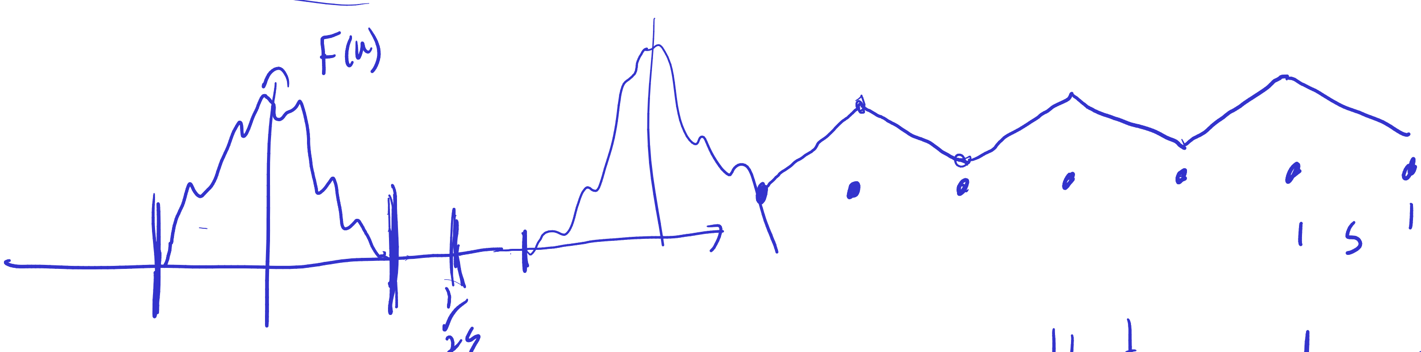
- Example: relation between space, sampling and frequency

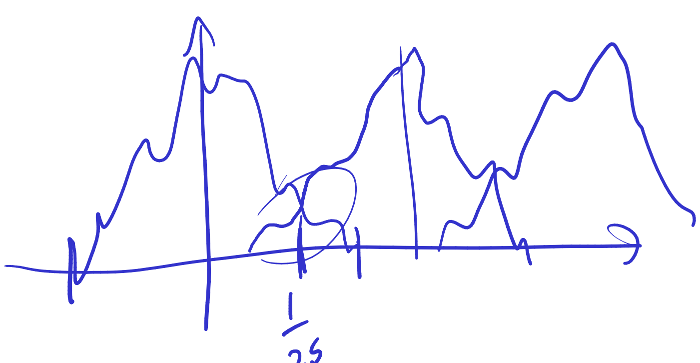


zero frequency component is in the top left corner output array.

The Nyquist-Shannon sampling theorem

"The largest frequency that can be represented in a signal sampled at intervals s is 1/2s"



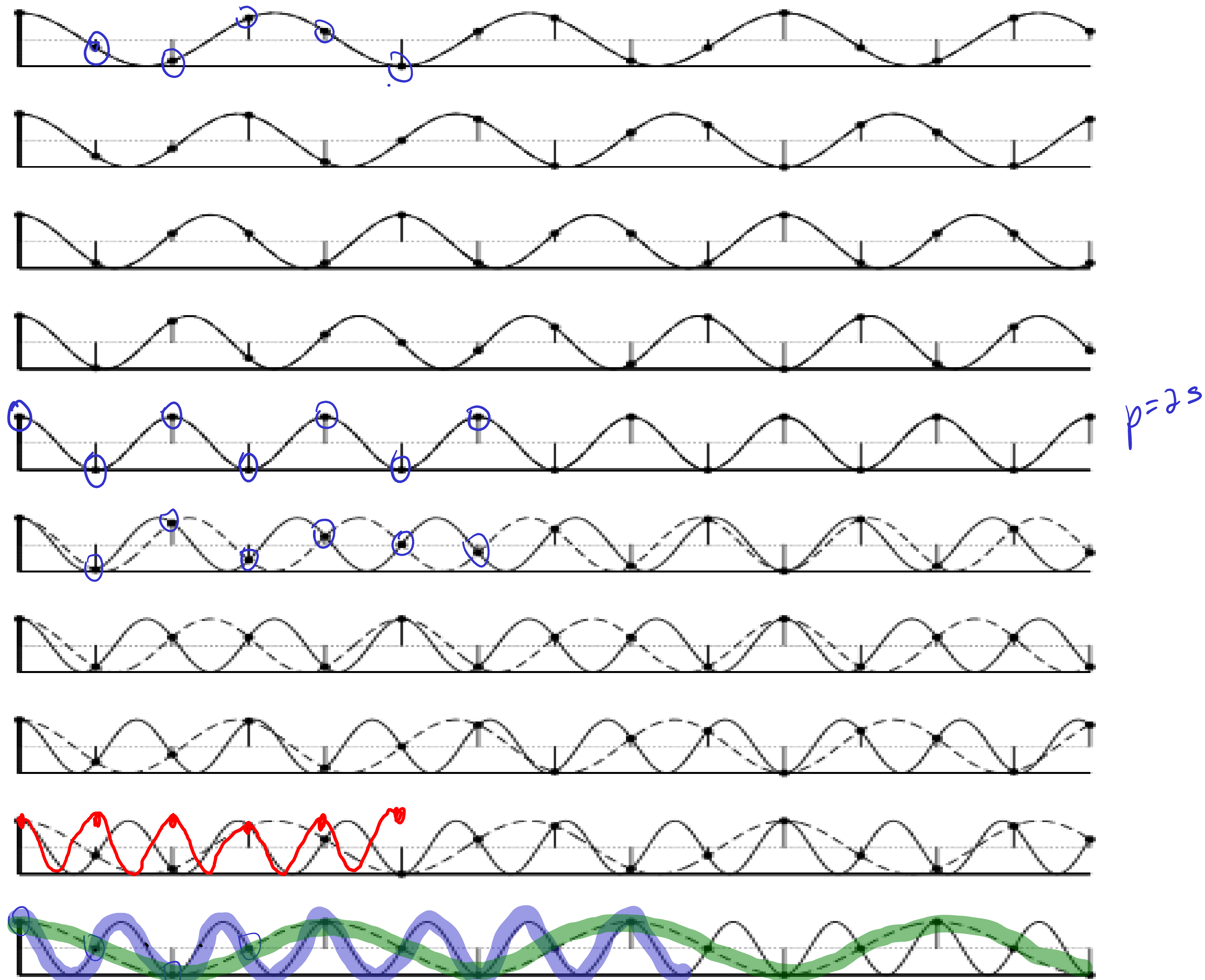


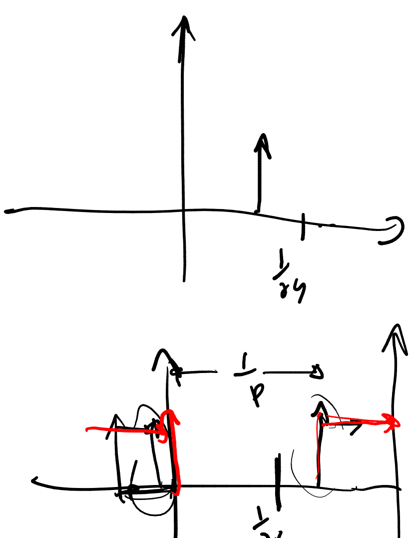
smallest period : 2s

$\Updownarrow$

largest spatial freq.: $\frac{1}{2s}$

Undersampling and aliasing

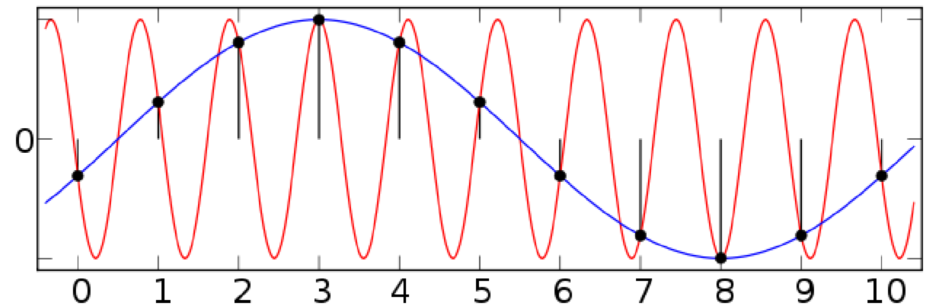




"aliasing": higher frequency is mapped onto a lower frequency

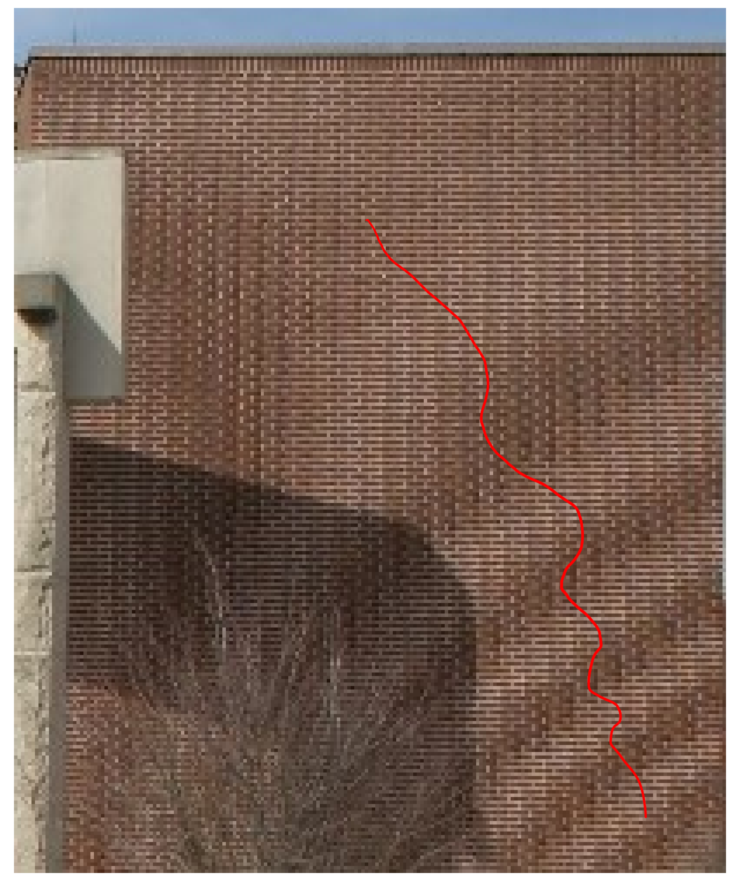
Aliasing

Moiré: after resampling, high spatial frequencies appear as low spatial frequencies



high resolution

low resolution

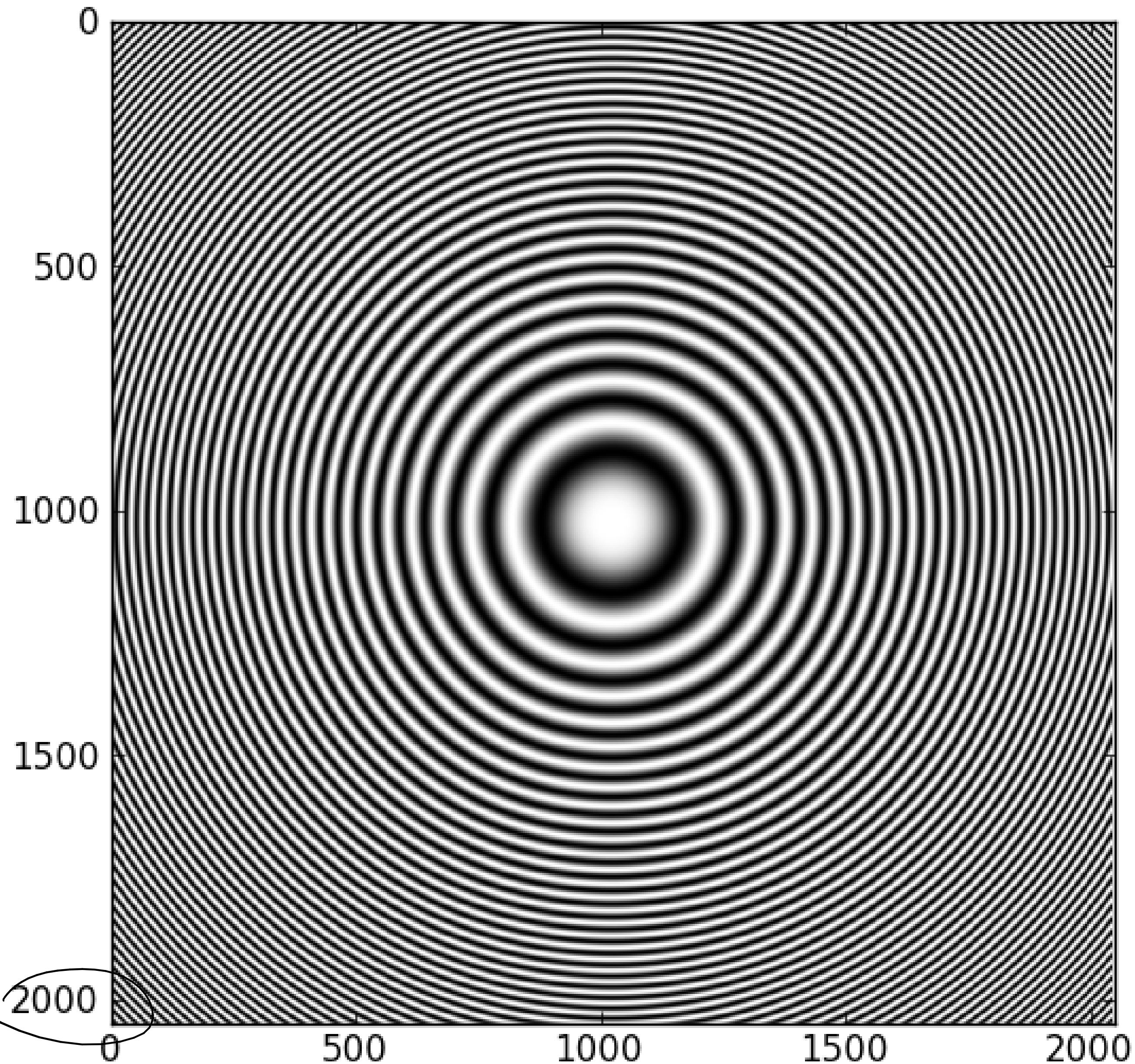


aliasing

source: http://wikipedia.org

Undersampling

"Fresnel zone" test pattern: radial linear increase in spatial frequency

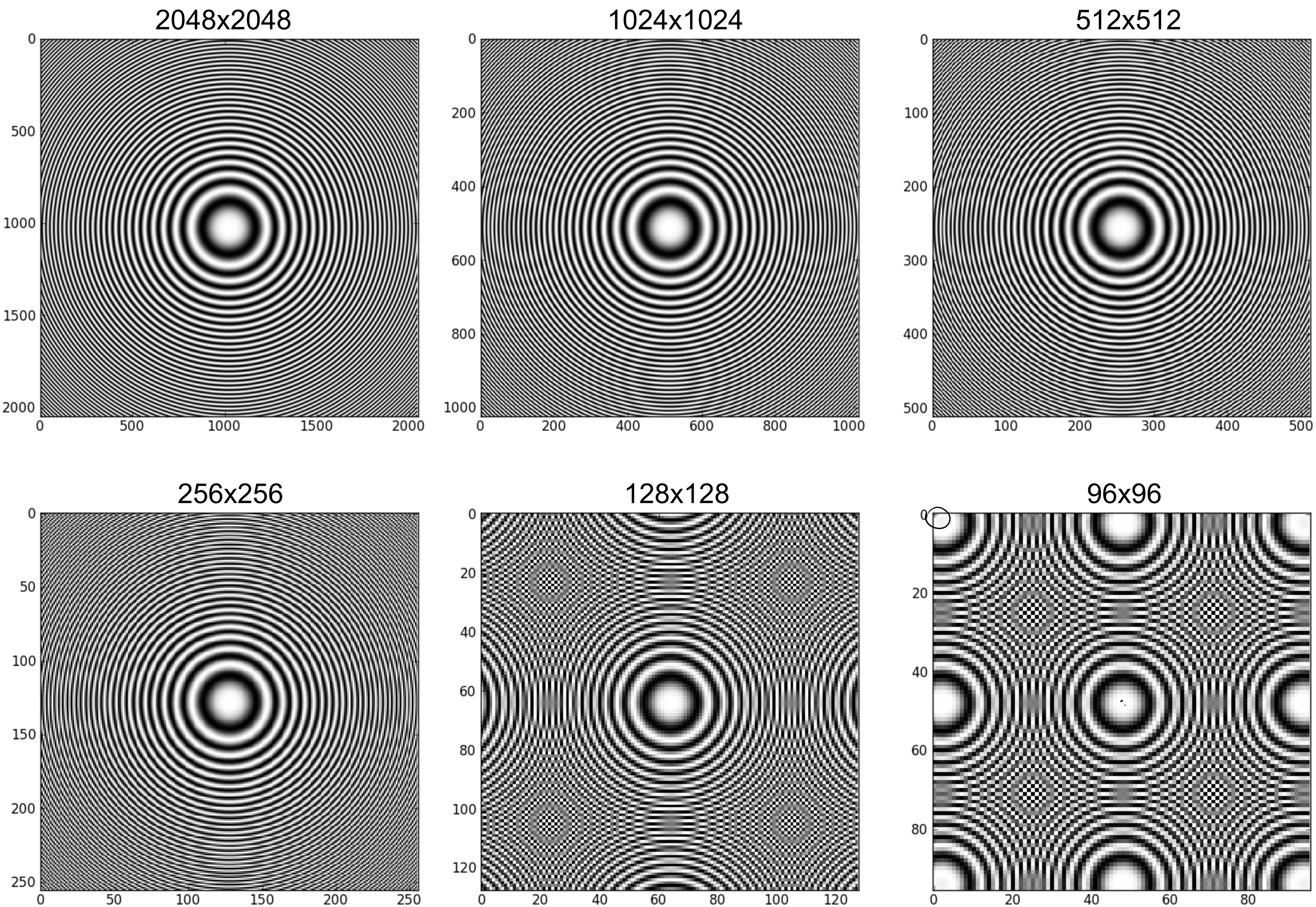


$\cos(r^2 \cdot a)$

$\cos(r \cdot (ra))$

frequency increases radially linearly with r

Undersampling & aliasing



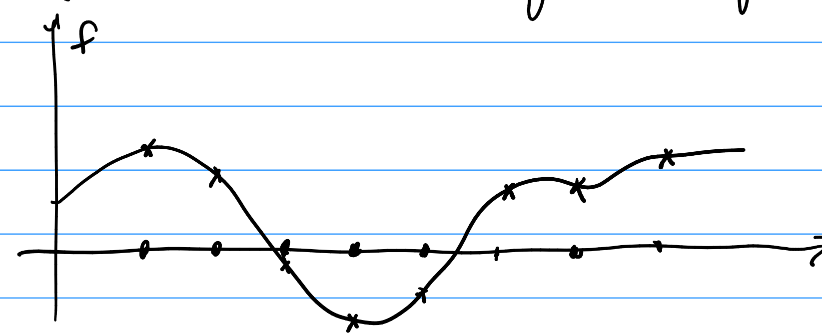
Sampling - summary

* what is sampling? The approximation of a signal (function) with a discrete set of values.

$f(x) \longrightarrow f_n = f(x_n)$ $\{f_n\}$ samples of f

$\{x_n\}$ sampling positions

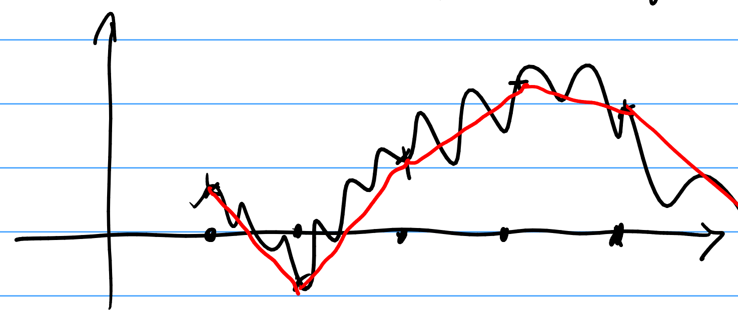
Most often $\{x_n\}$ is on a regular grid.



sampling is an unavoidable step in almost all data acquisition

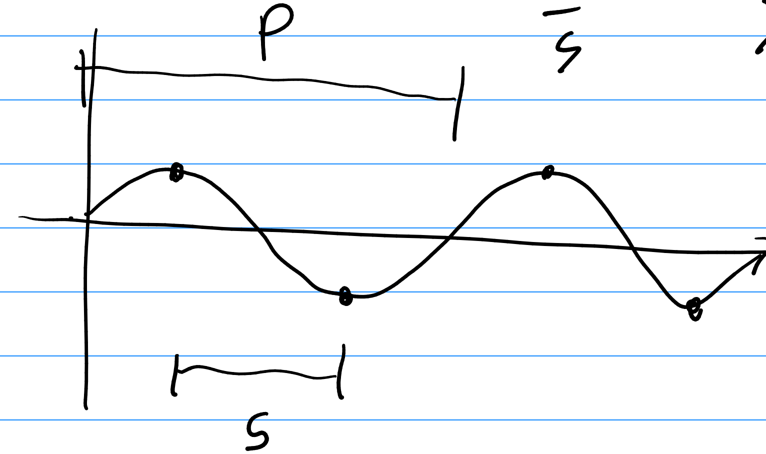
Can any signal be sampled - given a sampling rate?

No. Too high frequencies cannot be faithfully reproduced with a sampled signal.



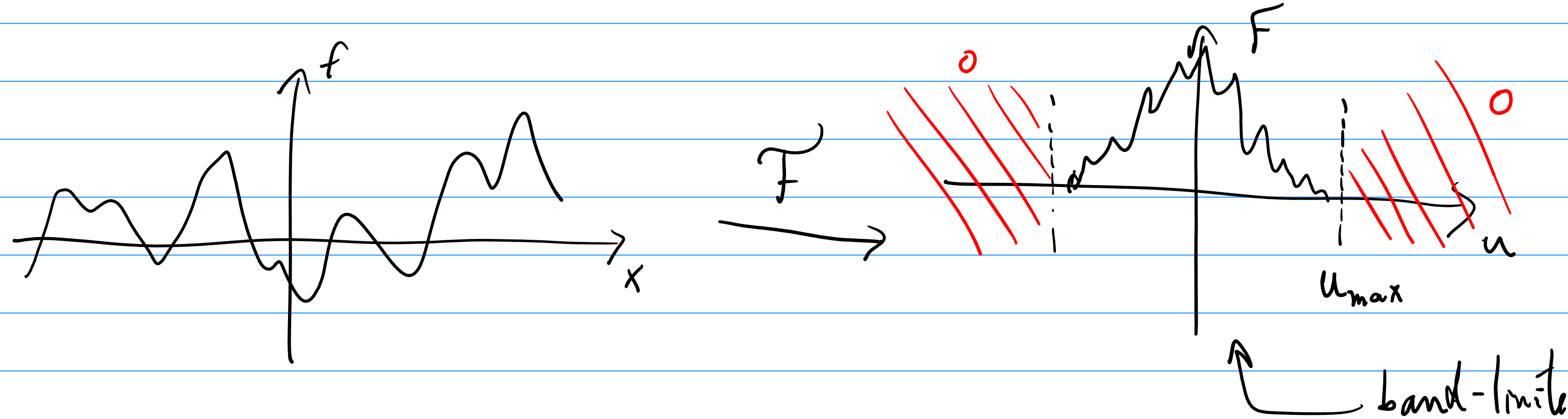
Condition: Nyquist theorem: sampling rate > 2 highest frequency

$\frac{1}{s} > \frac{2}{p}$



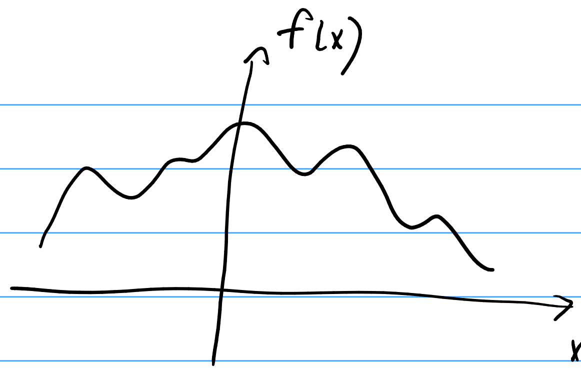
$(p > 2s)$

$\Rightarrow$ The only signals that can be faithfully reproduced by sampling are "band limited": they have a maximum frequency above which the signal is 0.

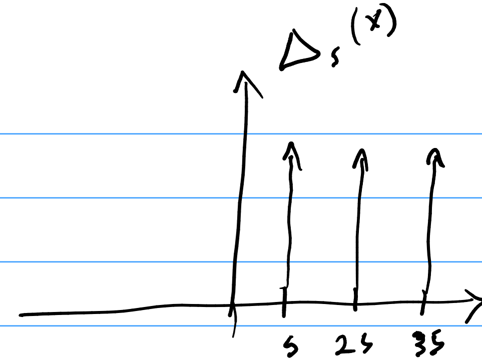


$\frac{1}{s} > 2u_{max}$ $\qquad$ $2u_{max}$ = Nyquist frequency

* What happens if $\frac{1}{s} < 2u_{max}$? Undersampling

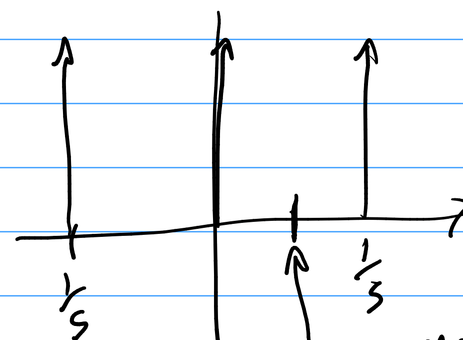


$\times$



$\downarrow \mathcal{F}$

$*$



$\rightarrow$

overlap $\Rightarrow$ aliasing

maximum frequency allowed for this sampling rate $= \frac{1}{2S}$

Fourier space translation

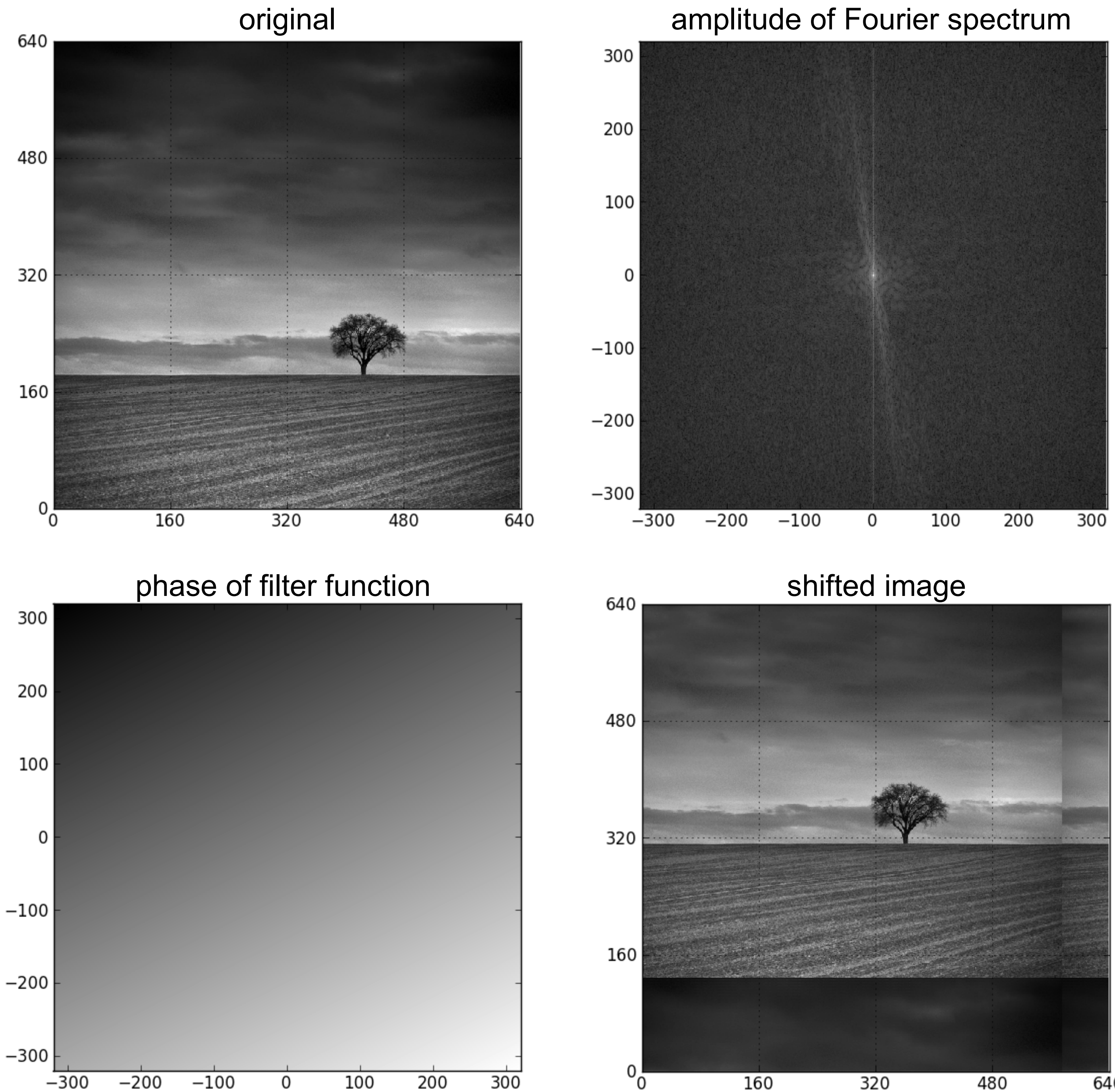


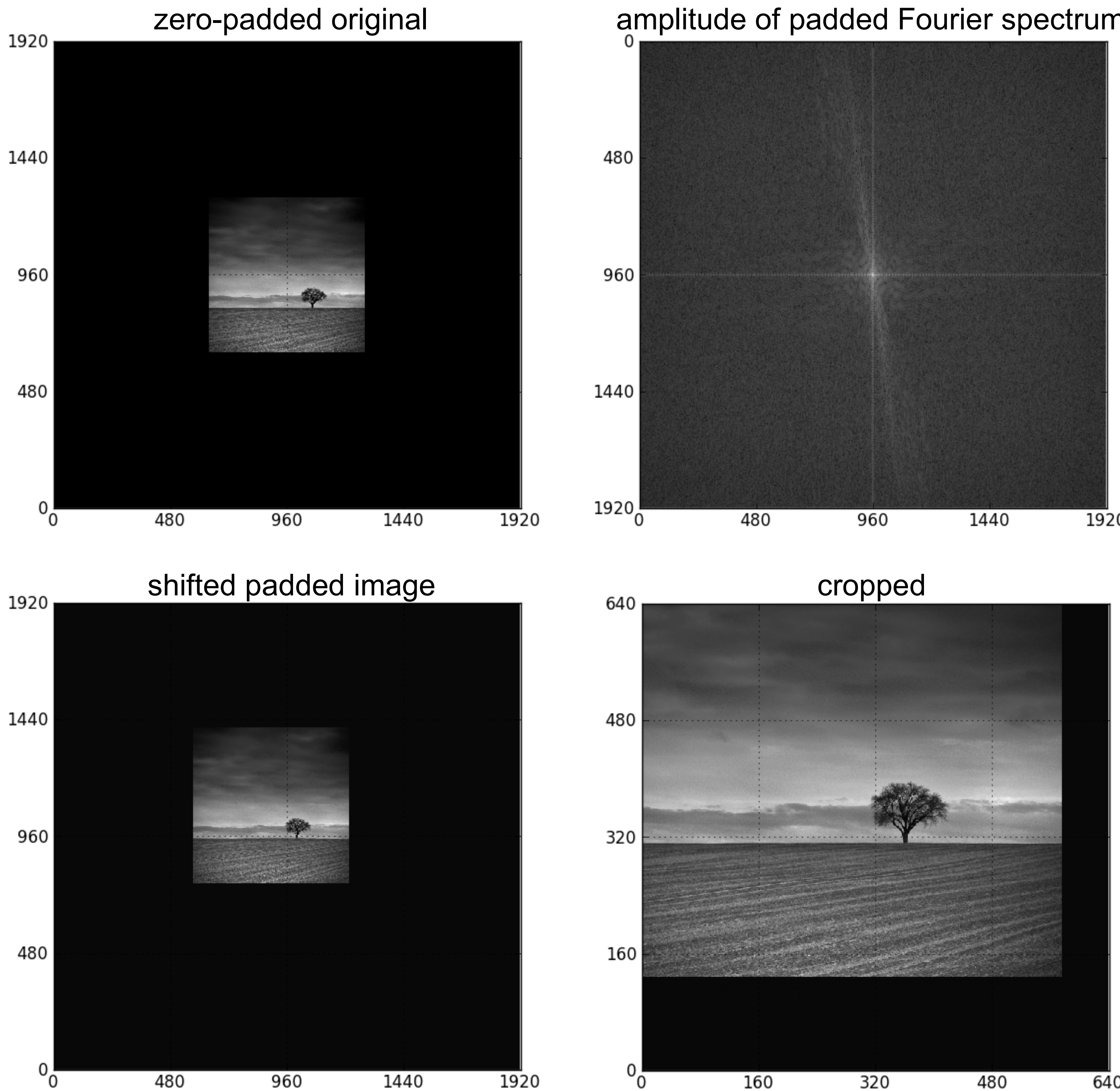
Image shifting using shifting property of FT

$$f(x+x_0) \xrightarrow{\mathcal{F}} F(u)\,e^{-2\pi i x_0 u}$$

$$\mathcal{F}\{f(\vec{r}+\vec{r}_0)\} = F(\vec{u})\,e^{-2\pi i \vec{r}_0\cdot\vec{u}}$$

Image gets wrapped around

Zero-padding

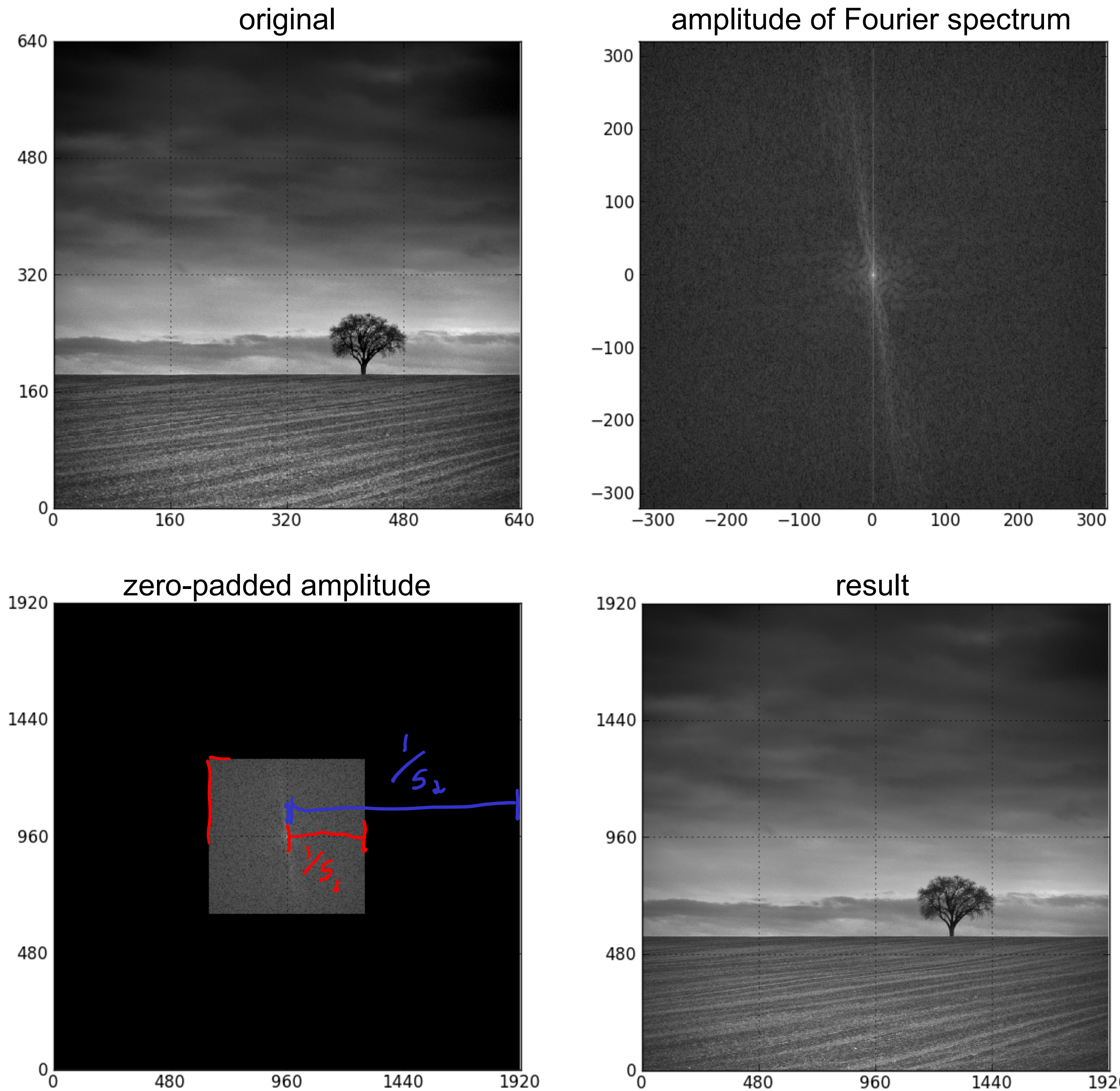


1. Add zeros around original image (*zero-padding*)

2. Shift using FT

3. Crop result

Zero-padding in Fourier space



Result: increased sampling!

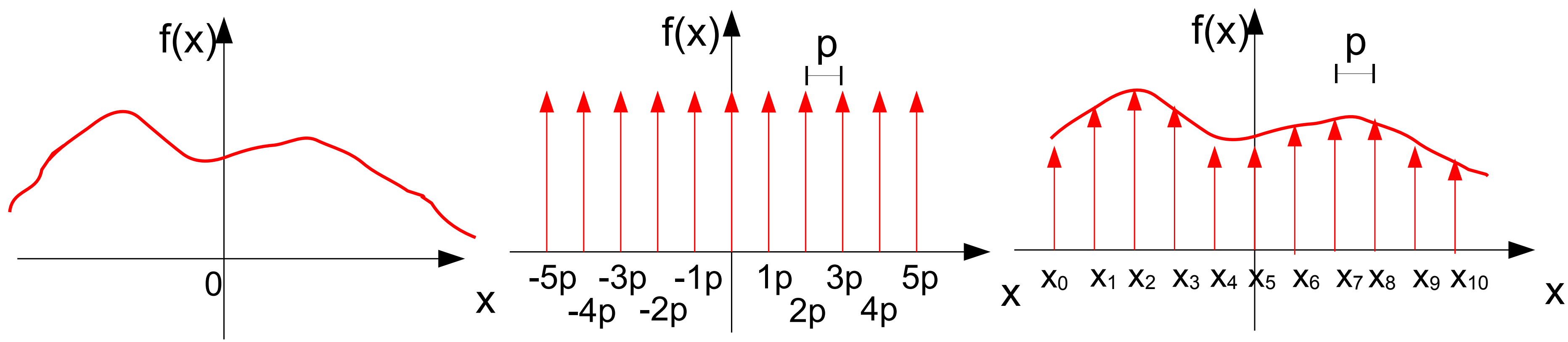
Is it magic? ← no. this is interpolation

$$\frac{1}{s_2} = 3\,\frac{1}{s_1}$$

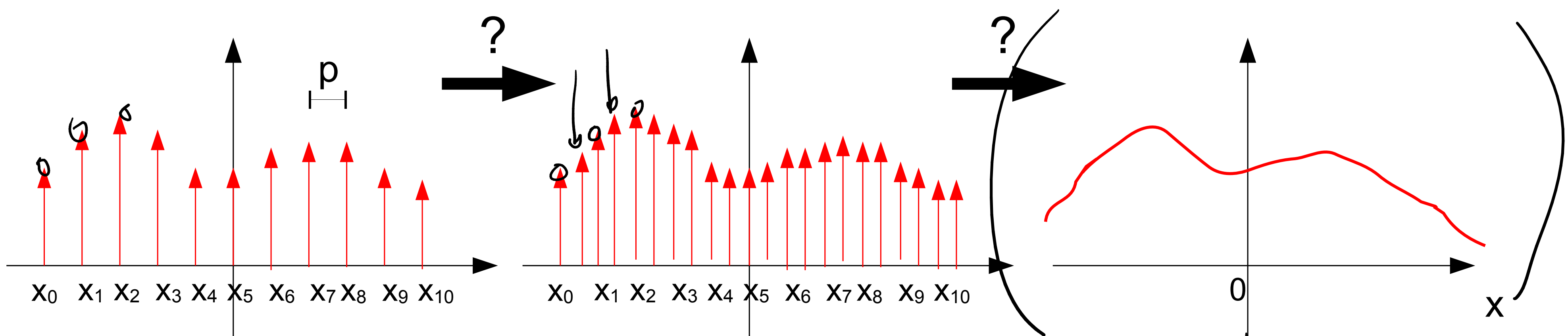
$$s_2 = \frac{s_1}{3}$$ ← pixels are $\frac{1}{3}$ of original

Interpolation

- Discrete sampling of a continuous function



- Reconstruct original function from sampled data?



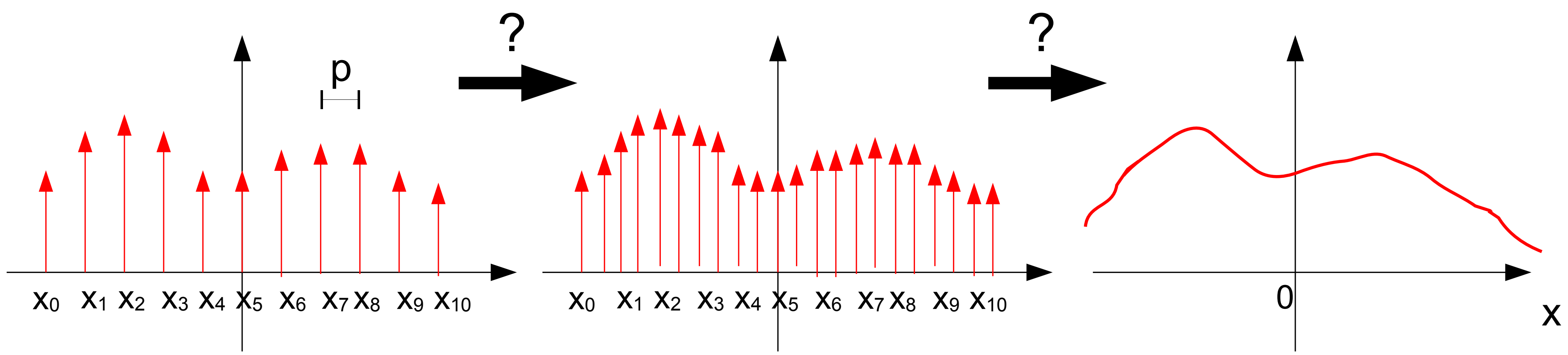
more often: resampling problem

ideal case

Interpolation

Finding unknown points between known ones

- wide field, many different approaches
- closely related to approximation theory and curve fitting



Interpolation

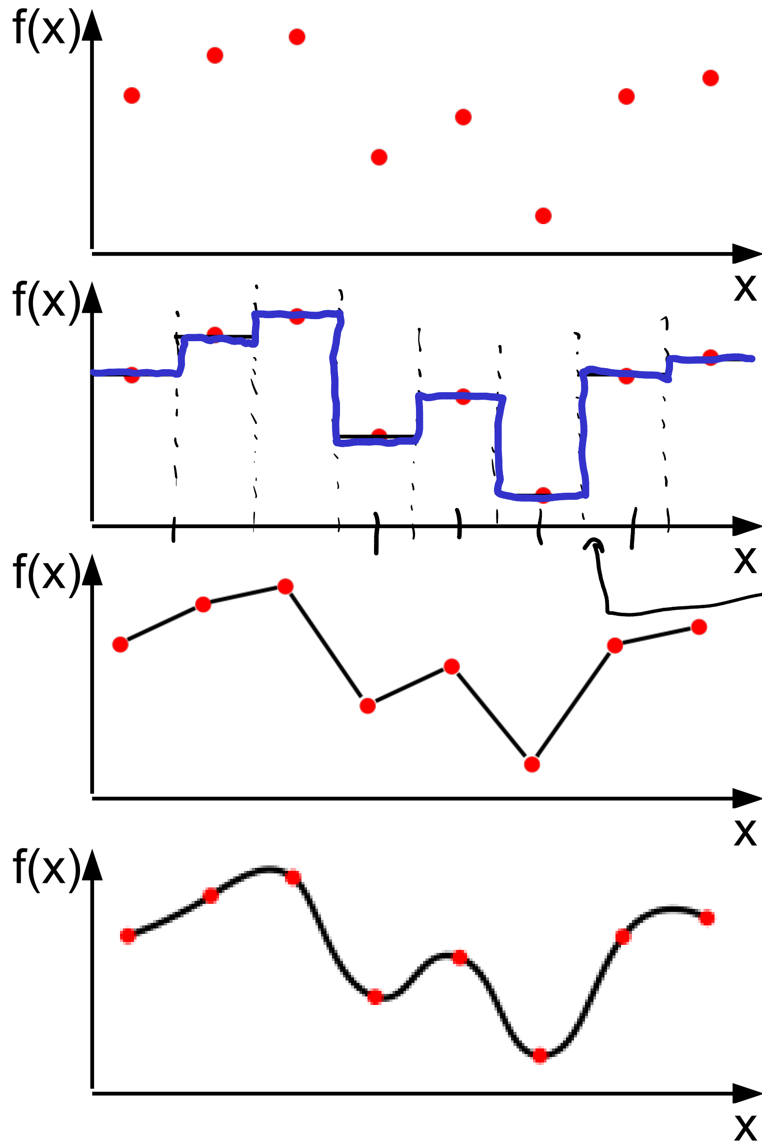
Various "classical" interpolation methods available

sampled points

nearest neighbor

linear

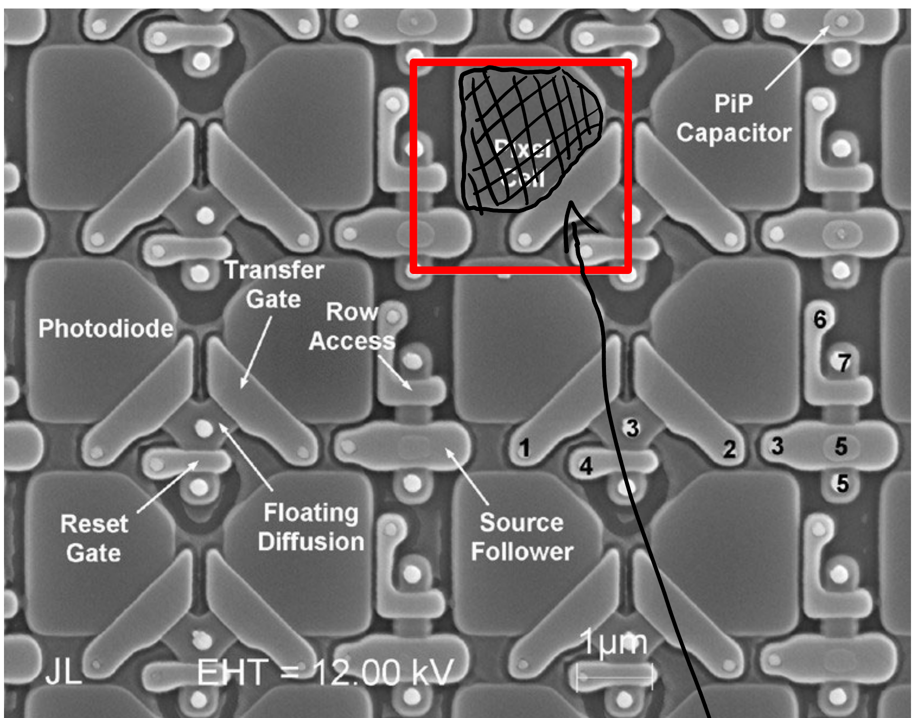
cubic spline



sharp transitions midway between samples

Pixels

- distinguish between detector pixels, image pixels and screen pixels
  - detector pixels are rarely square
  - image pixels are commonly, but not necessarily square
  - screen pixels are rarely square



Detector pixel

region sensitive to light

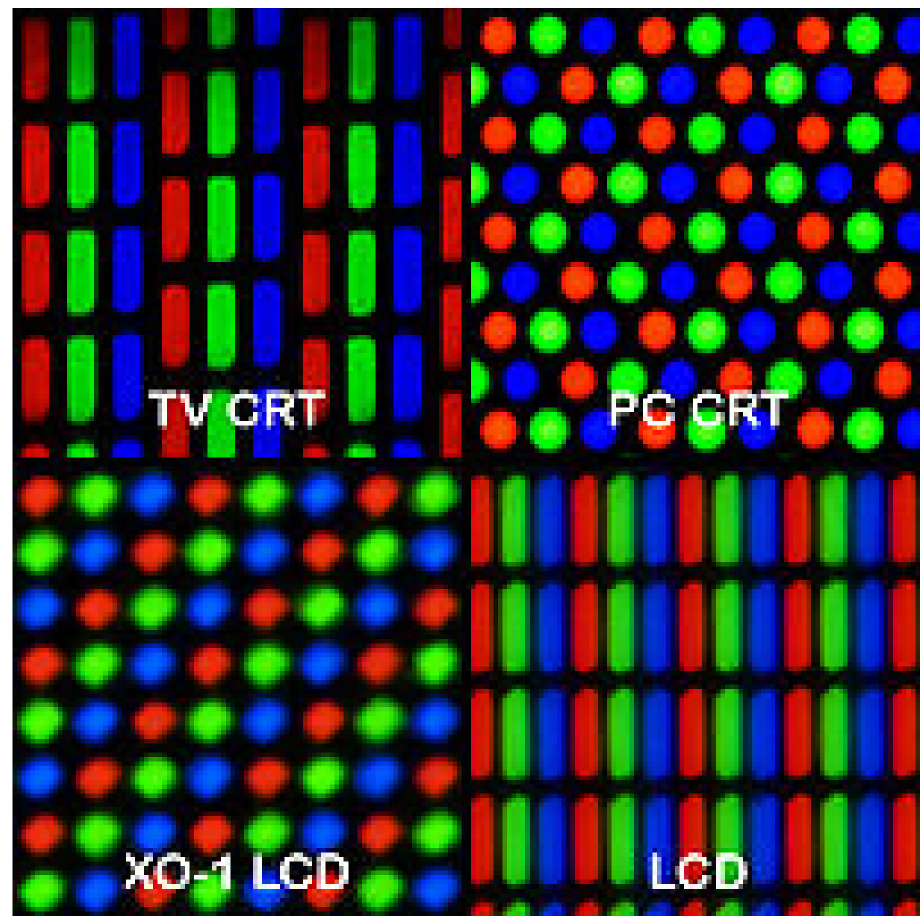
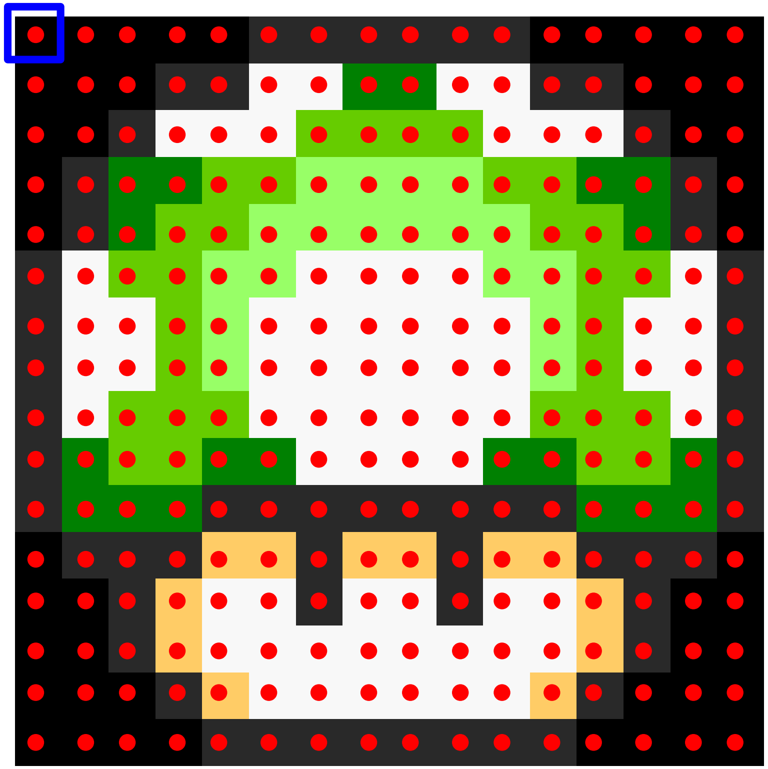


Image pixels

Images are discrete samples of a continuous function

- ...with coordinates
- ...and values (voltage at coordinate, integral over pixel area, ...)
- ...represented by <u>pixel basis functions</u> on a <u>sampling grid</u>



Linear interpolation

- Interpolation as an operator

$$f(x) = \mathcal{L}\{f_n\}$$

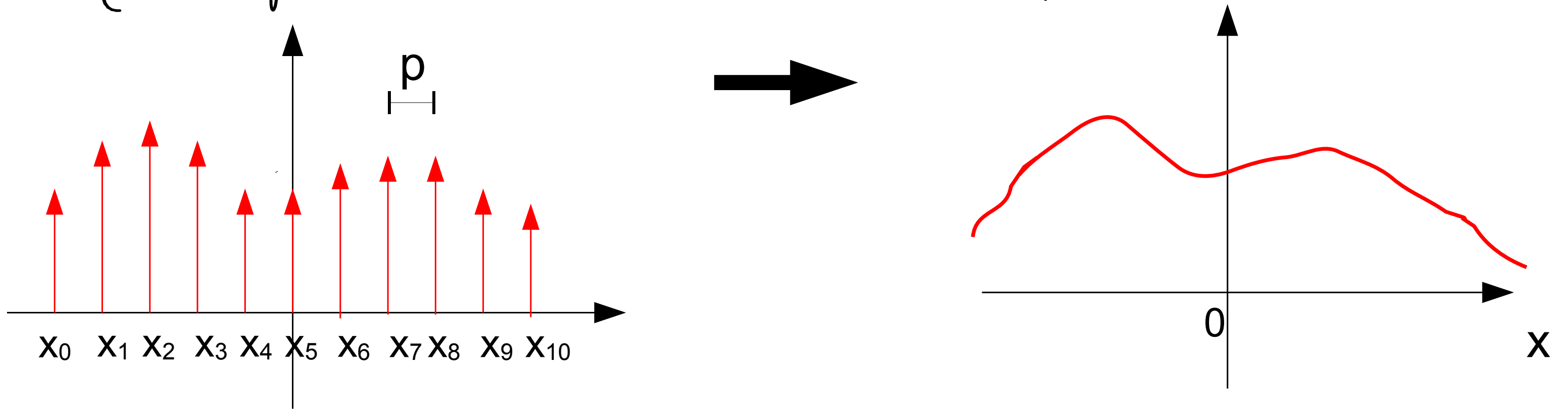
- Linearity of interpolation operator

$$\mathcal{L}\{f_n + g_n\} = \mathcal{L}\{f_n\} + \mathcal{L}\{g_n\}$$

- Shift invariance

$$\mathcal{L}\{f_{n+n_0}\} = f(x + n_0 s)$$

- Kernel (interpolation seen as a convolution)

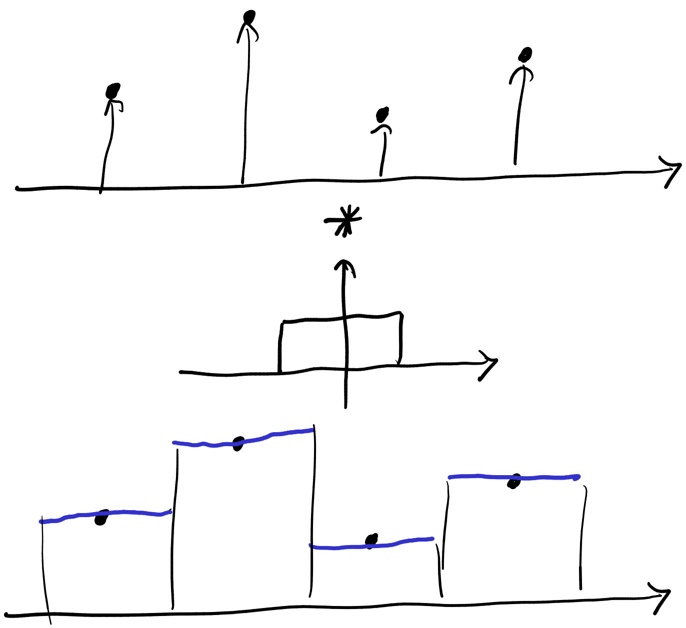


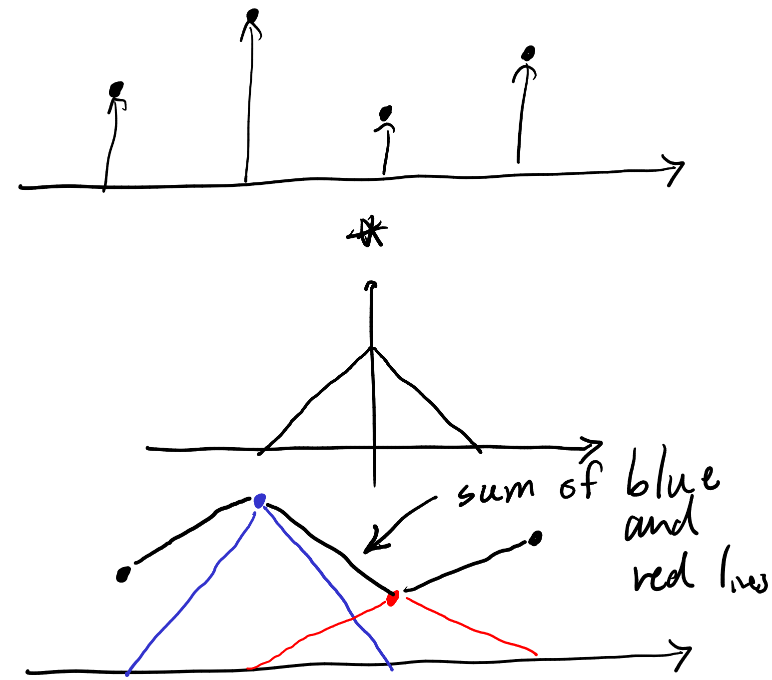
Linear interpolation

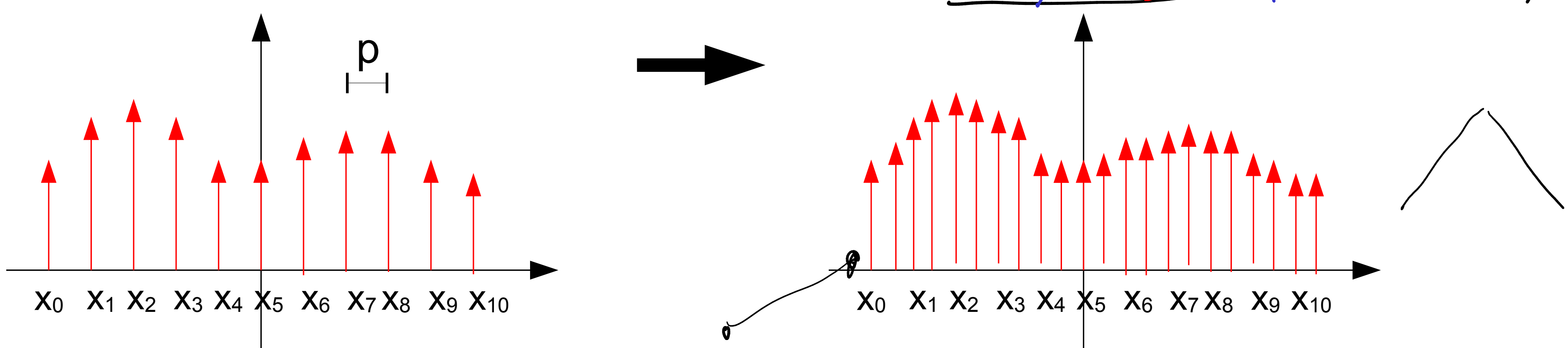
- Linear interpolation can be written as a convolution with a kernel (e.g. a basis function)

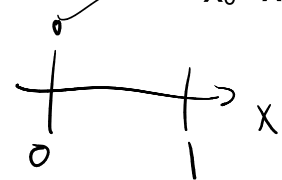
nearest neighbor = convolution with box function ("rect" function in 1D)

linear interpolation





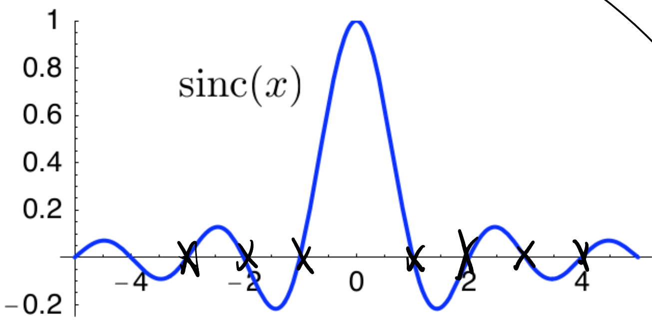




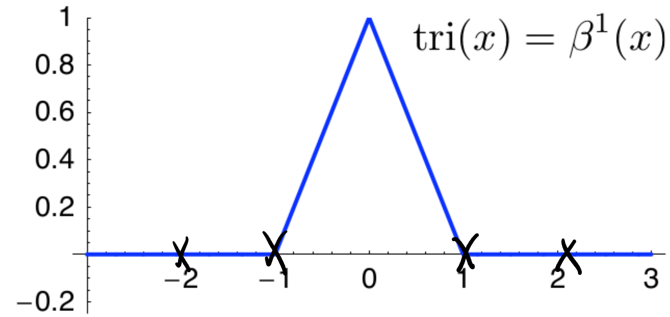
$f_0(x-1) + f_1 x$

Linear interpolation

- Bandlimited

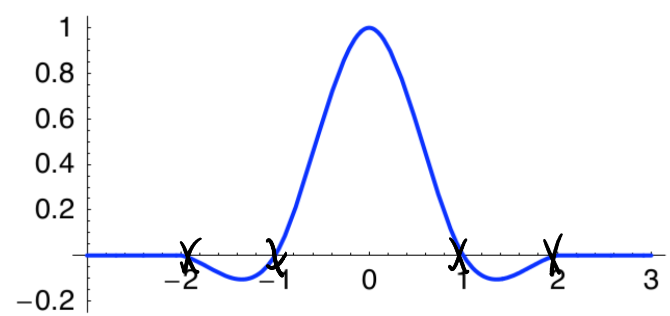


- Piecewise linear



tri x tri

- Cubic convolution



Interpolation condition:

$$\varphi_{\mathrm{int}}(k) = \delta_k = \begin{cases} 1, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

[Keys, 1981; Karup-King 1899]

source: http://bigwww.epfl.ch/tutorials/unser_isbi_06_part1

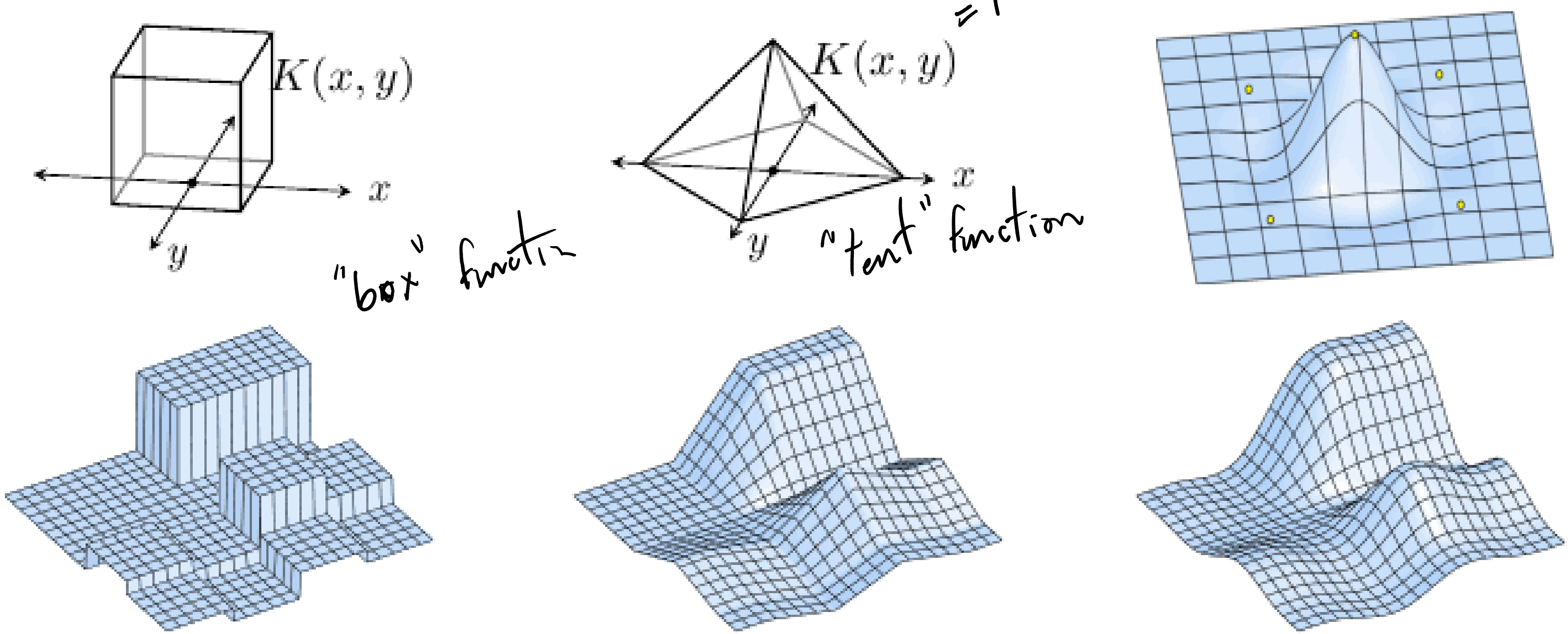
Interpolation via convolution

2D interpolation

- Make 2D interpolation linear in each variable

nearest neighbor | bilinear | bicubic

$= K(x)K(y)$

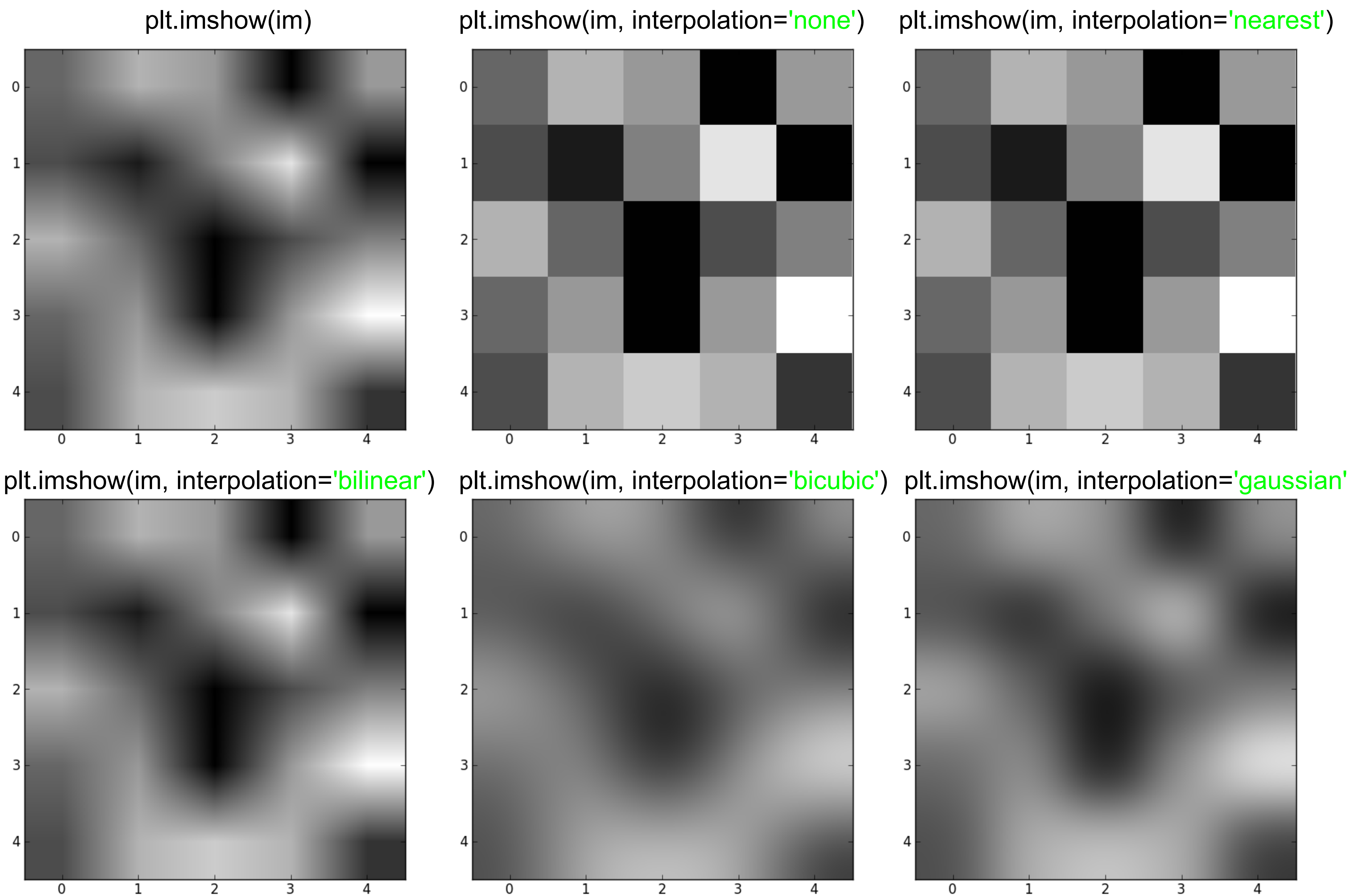


"box" function

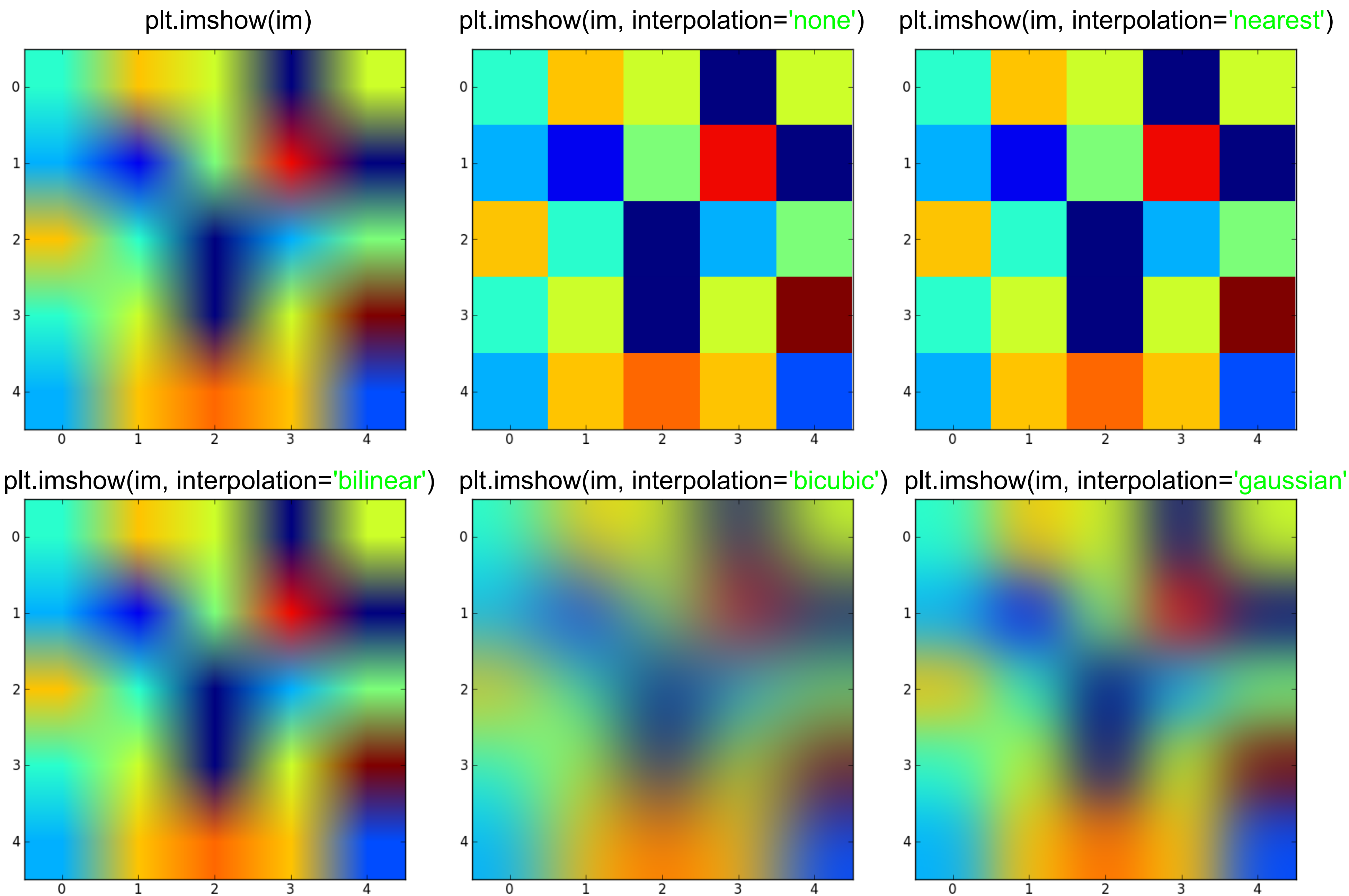
"tent" function

source: http://www.ipol.im/pub/art/2011/g_lmii/

Python plotting

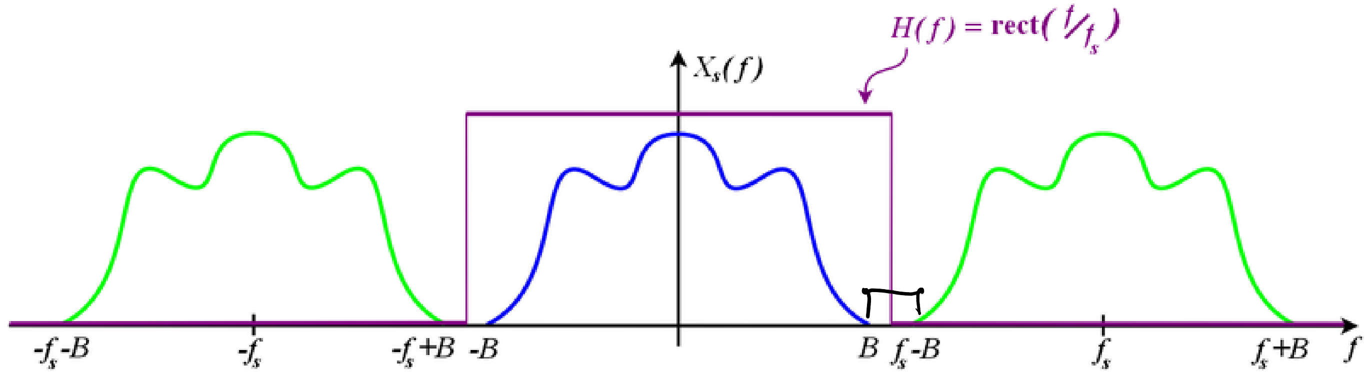


Python plotting

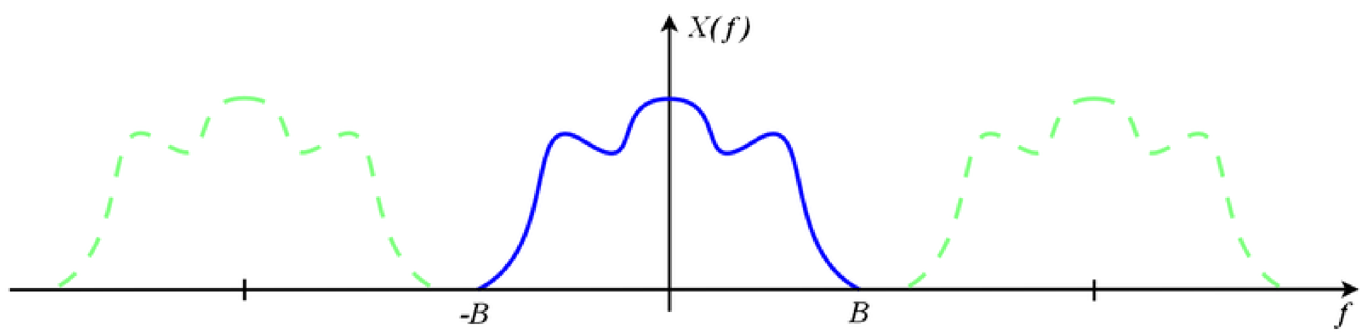


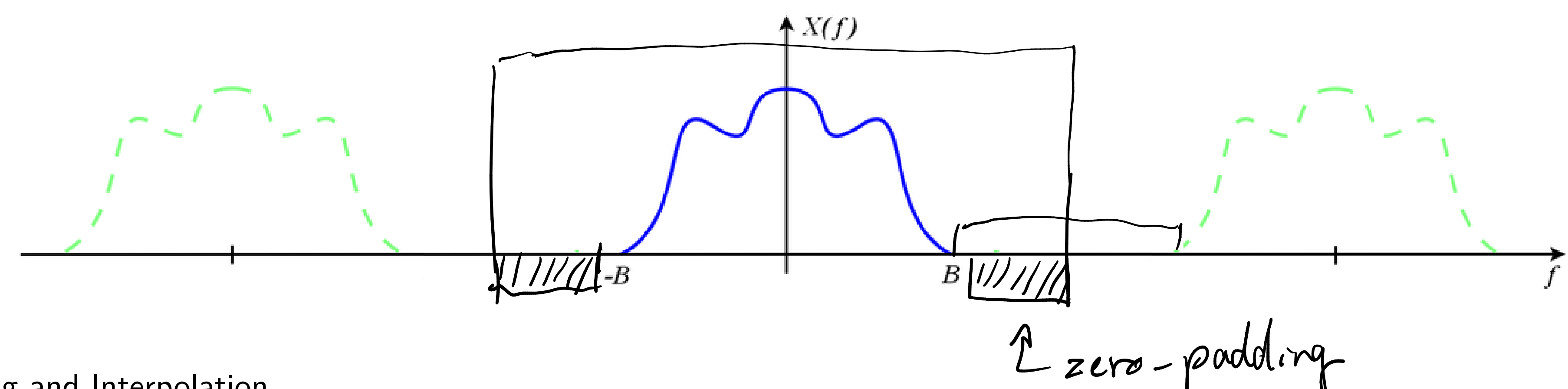
Sinc interpolation and zero-padding

Also known as "Whittaker–Shannon interpolation"



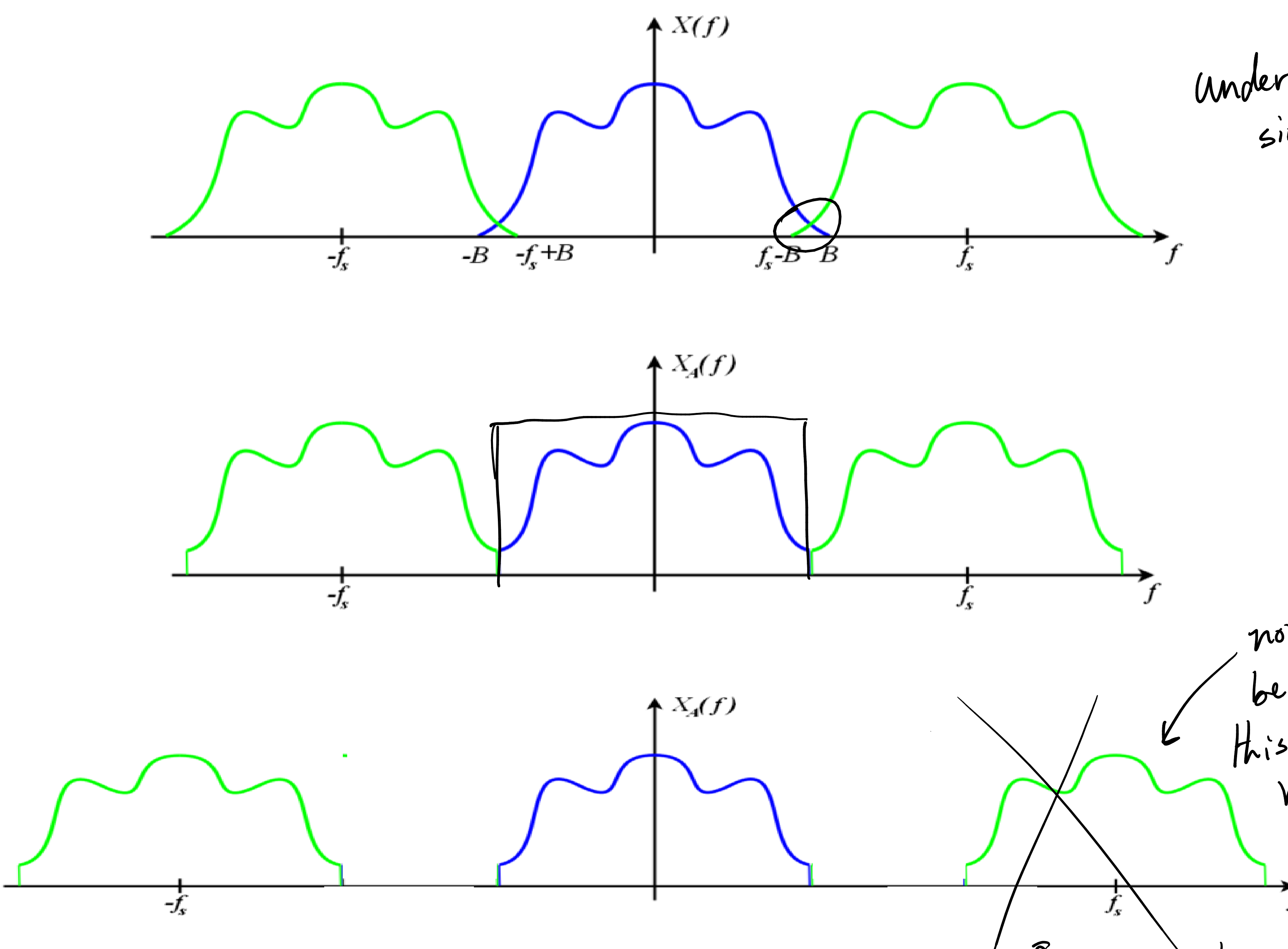
(Fourier space)





Sinc interpolation and zero-padding

Also known as “Whittaker–Shannon interpolation”



undersampled signal

non-zero because this is a resampling operation

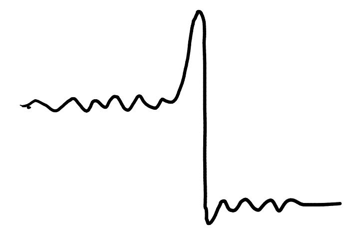
← if interpolation

Reconstruction from samples

- Sinc interpolation can perfectly reconstruct a function from its samples if

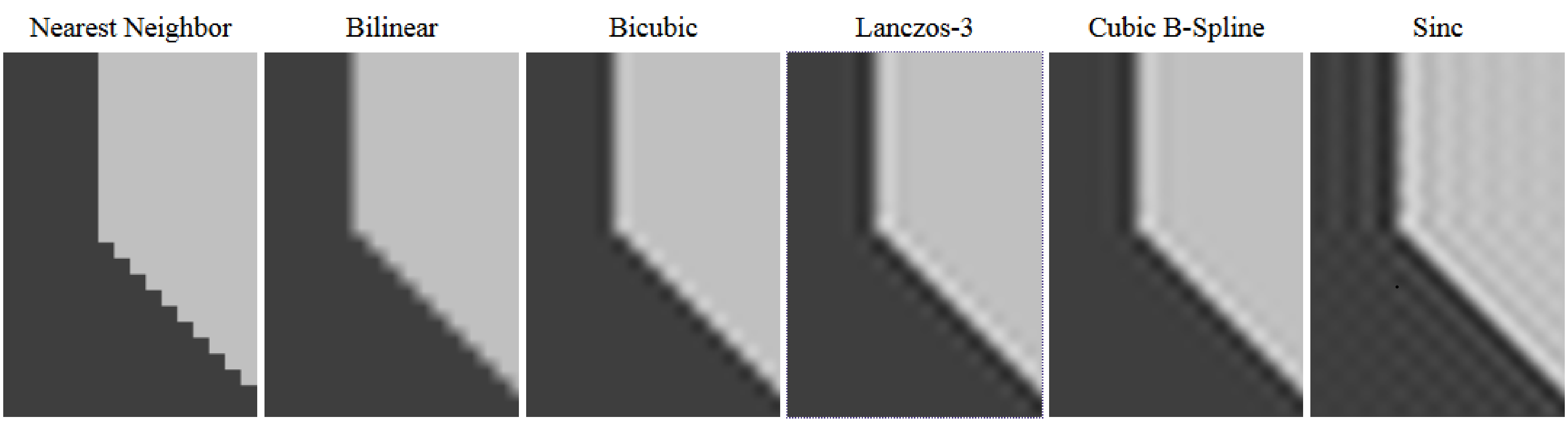
equivalent

  - sampled at a rate higher than Nyquist rate
  - bandlimited up to Nyquist frequency
  - no aliasing



related to "ringing artifact"
"Gibbs phenomenon"

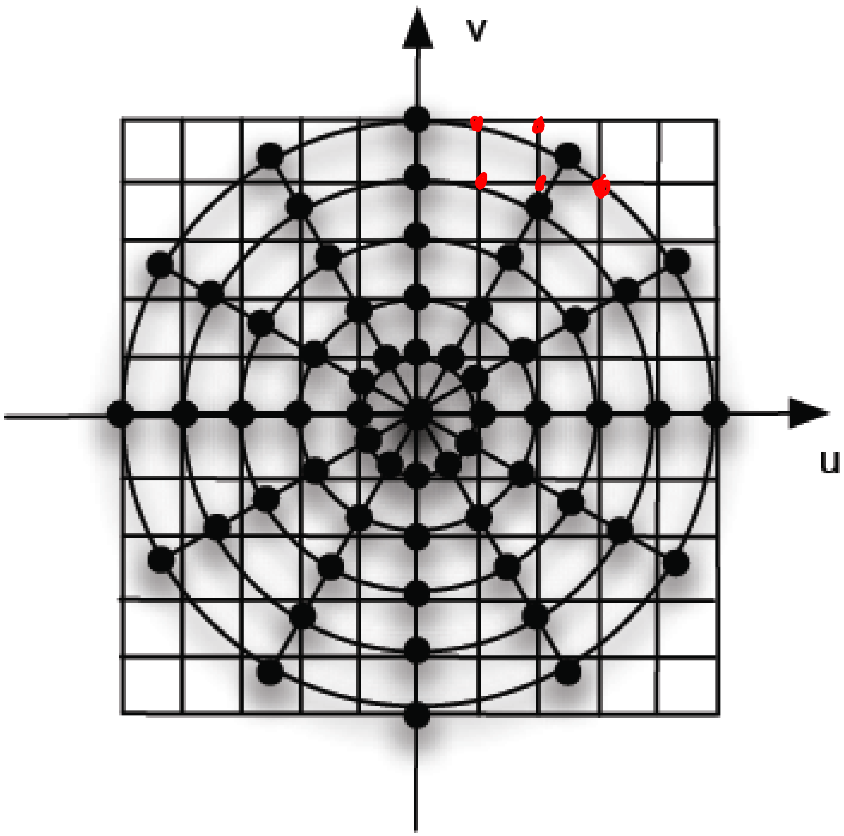
- Sinc interpolation introduces ringing otherwise, due to leakage of aliased frequencies



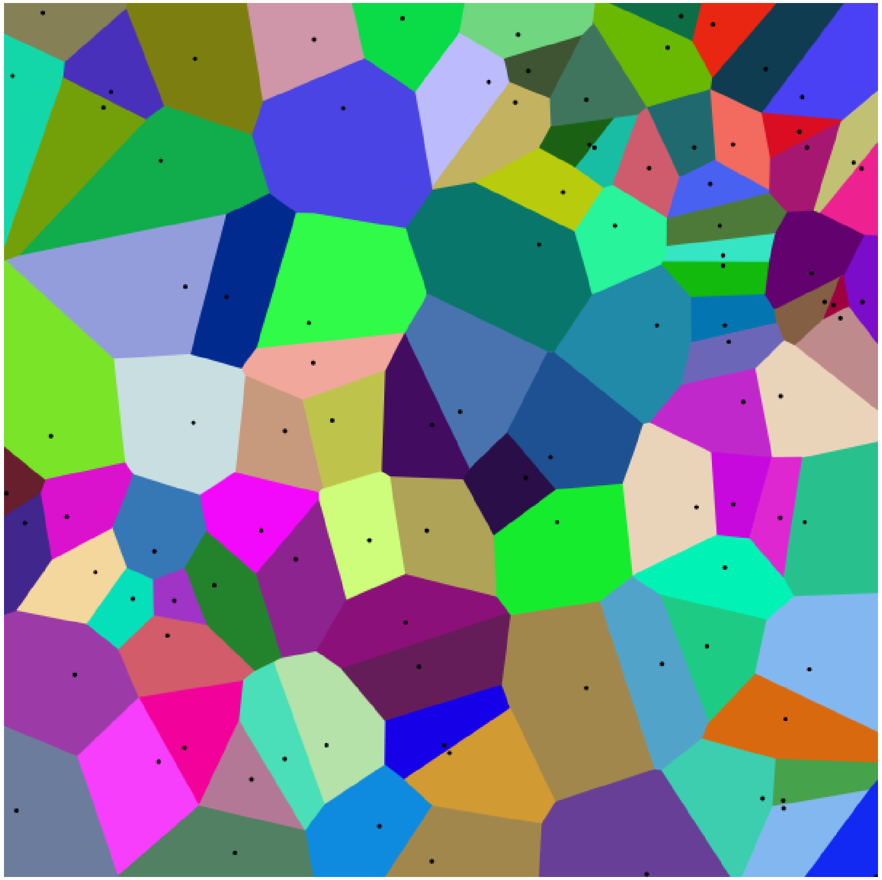
Linear interpolation of a step edge: a balance between staircase artifacts and ripples.

Other Interpolation

- Change from polar to cartesian grid
- Linear, but not translation invariant



polar vs. cartesian sampling



irregular sampling

Summary

- Images can be represented as a sampling grid and pixel basis functions
- Need for interpolation arises when changing the grid
- Linear and translation invariant interpolation can be written as a convolution with an interpolation kernel function
- Typical interpolation kernels include nearest neighbor, linear, cubic and higher B-spline interpolation
- Zero-padding in one domain equals sinc interpolation in the other
- "ideal" sinc interpolation may lead to ringing artifacts