THETA TERM

In YM theory, one has kinetic term

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \quad \leftarrow \quad \text{Lorentz \& gauge inv. with at most two derivatives}$$

There is another term that satisfies these conditions:

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \qquad \text{with} \quad \tilde{F}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$$

→ This term violates CP. It can be phenomenologically important!
↓

Under CP

$$A^a_0(t,\bar{x}) \mapsto -A^a_0(t,-\bar{x}) \qquad \partial_0 \mapsto \partial_0$$

$$A^a_i(t,\bar{x}) \mapsto A^a_i(t,-\bar{x}) \qquad \partial_i \mapsto -\partial_i$$

Hence CP and P are both violated, then one can have the following terms in the Lagrangian

$$\mathcal{L}_{CPV} = \theta_{QCD} \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \underbrace{F^a_{\mu\nu} F^a_{\alpha\beta}}_{SU(3)} + \theta_2 \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} \underbrace{W^a_{\mu\nu} W^a_{\alpha\beta}}_{SU(2)}$$

$$+ \theta_1 \frac{g'^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} \underbrace{B_{\mu\nu} B_{\alpha\beta}}_{U(1)}$$

θ - terms

In YM θ parametrizes physically inequivalent theories, despite the fact that $\mathcal{L}_\theta$ is a total derivative

$$\frac{\theta}{64\pi} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \, tr(\mathbb{F}_{\mu\nu} \mathbb{F}_{\rho\sigma})$$

$$= \frac{\theta \epsilon^{\mu\nu\rho\sigma}}{8} tr\left(\partial_\mu A_\nu \partial_\rho A_\sigma + 2i A_\mu A_\nu \partial_\rho A_\sigma \right)$$

$$= \frac{\theta}{8} \partial_\rho \left\{ \epsilon^{\mu\nu\rho\sigma} tr\left(A_\sigma \partial_\mu A_\nu + \frac{2i}{3} A_\sigma A_\mu A_\nu \right) \right\} \quad (*)$$

(*) $\Rightarrow$ it does not modify the equations of motion.

However in quantum theory, when π^1(CONFIG. SPACE) is <u>non-trivial</u>, then total derivatives matter.

For non-ab. gauge theories π^1(CONFIG.SP.) $= \mathbb{Z}$.

(*) $\Rightarrow$ it does not contribute to perturb. theory: total derivative is $\propto \sum p_i^{\text{external}}$

However, in presence of fermions charged under G, θ is unphysical because it depends on how the phase of those fermions is chosen.

To understand this effect let us do a REDEFINITION of all fermions:

$$\psi_I \longmapsto e^{i\beta_I \gamma_5}\psi_I$$

I is a FLAVOUR INDEX (For QCD $I=1,\dots,6$)
β_I are real phases

$$\Rightarrow \mathcal{D}\psi\mathcal{D}\bar\psi \mapsto e^{-\frac{i}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma} F_\alpha^{\mu\nu} F_\alpha^{\rho\sigma} \sum_I \beta_I} \mathcal{D}\psi\mathcal{D}\bar\psi$$

Axial anomaly

$\rightsquigarrow$ This is equivalent to

$$\theta \mapsto \theta - 2\sum_I \beta_I$$

$\Rightarrow$ Only the combination $\theta - \arg\det(Y_u Y_d)$ is physical (i.e. invariant under field redefinition):

$I = u, d \times 3$

$I = (\alpha, i)$ $i = u, d$ $\alpha, \beta = 1,2,3$

$$\sum_{\alpha,\beta} Y_u^{\alpha\beta}\, \bar q_L^\alpha \tilde\Phi u_R^\beta \mapsto \sum_{\alpha,\beta} e^{-i(\beta_\alpha^u + \beta_\beta^u)} Y_u^{\alpha\beta}\, \bar q_L^\alpha \tilde\Phi u_R^\beta$$

$\Rightarrow$ phase of $Y_u^{\alpha\beta}$ shifted by $-(\beta_\alpha^u + \beta_\beta^u)$
(i.e. row α multiplied by $e^{-i\beta_\alpha^u}$, column β multiplied by β_β^u)

$$\Rightarrow \det Y_u \to e^{-2i\sum_\alpha \beta_\alpha} \det Y_u$$

Analogously $\det Y_d \to e^{-2i\sum_\alpha \beta_\alpha^d} \det Y_d$

$$\Rightarrow \arg\det Y_u Y_d \mapsto \arg\det Y_u Y_d - 2i \sum_{\substack{\alpha=1,2,3 \\ i=u,d}} \beta_\alpha^i$$

$\to$ the same shift as Θ !

Moreover, thanks to axial anomaly, we can show that Θ and $\Theta + 2\pi$ are physically indistinguishable

In fact, we can always absorb the 2π shift by rotating one ψ_I by $\beta_I = \pi$, that does not affect the Yukawa matrices

$\downarrow$

This can also proved by noticing that $S_\Theta = \Theta \cdot n \quad n \in \mathbb{Z}$

$\Rightarrow e^{iS_\Theta} \to e^{iS_\Theta}$ by taking $\Theta \mapsto \Theta + 2\pi$.

$\uparrow$ instanton number (topological quantity)

Even if θ-term does not contribute to any order in perturbation theory, nevertheless several SM observables have a non-trivial θ-dependence, such as

- vacuum energy;
- electric dipole moment of the neutron;
- pion mass & proton-neutron mass splitting.

This dependence can be understood qualitatively as the contribution to the SM P.I. of gauge config. with non-vanishing topological charge $Q = \frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu}$.

Example: VACUUM ENERGY of QCD $E(\theta)$.

- $E(\theta)$ is defined as
$$e^{-V_4 E(\theta)} = \int D\phi\, e^{-S[\phi] - i\frac{\theta}{32\pi^2}\int F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}}$$

- $E(\theta+2\pi) = E(\theta)$
- $E(-\theta) = E(\theta)$: do field redef. $\phi \to \phi^{CP}$ and use $D\phi^{CP} = D\phi$ and $S[\phi^{CP}] = S[\phi]$ for QCD

- $\theta = 0$ is the absolute min. of $E(\theta)$ as
$$e^{-V_4 E(\theta)} = \left| e^{-V_4 E(\theta)} \right| \leq \int D\phi \left| e^{-S[\phi] - i\frac{\theta}{32\pi^2}\int F\tilde{F}} \right| =$$
$$= \int D\phi\, e^{-S[\phi]} = e^{-V_4 E(0)}$$

$\Rightarrow \quad E(\theta) \geq E(0)$

$\Rightarrow$ parity sim. $\theta \to -\theta$ is not spont. broken. (Vafa-Witten theorem)

- Let us extract the form of $E(\theta)$ and show that $E(\theta) \neq 0$.

To do this we need some approximation method to compute the P.I.

⇝ use saddle pt approximation :

- take P.I. in Euclidean formulation as we did above ;
- contribution of integrand mostly given by config. with FINITE ACTION and that MINIMIZE $S[\phi]$;
- expand $S[\phi] = S[\phi_{min}] + \frac{1}{2} \frac{\delta^2 S}{\delta \phi^2}\Big|_{\phi_{min}} \delta\phi^2 + \dots$

  it goes out of P.I. — Gaussian integral

⇝ In YM (or QCD) config. A_μ^a that MINIMIZE the Eucl. action and have FINITE ACTION are called INSTANTONS :

- Finite action $\Rightarrow$ $F_{\mu\nu}^a \xrightarrow[|x| \to \infty]{} 0$

  $\Rightarrow$ $A_\mu^a \xrightarrow[|x| \to \infty]{}$ PURE GAUGE, i.e. $i \partial_\mu U(\hat{x}) U(\hat{x})^{-1}$ for some $U(\hat{x}) : S^3_\infty \to G$ ($\hat{x} \in S^3_\infty$)

- Instantons are classified by maps $U : S^3 \to G$; these are classified by HOMOTOPY CLASSES (classes are done by maps that can be CONTINUOUSLY deformed into each other) in $\pi_3(G) = \mathbb{Z}$. The classes are labelled by an INTEGER, that is precisely

$$n = \frac{1}{32\pi^2} \int d^4x \, F_{\mu\nu}^a \tilde{F}^{a\mu\nu} \qquad \text{(INSTANTON NUMBER)}$$

- Maps that have the same inst. numb. have the same Θ-term : $S_\Theta = \Theta n$.

$$e^{-V_4 E(\Theta)} = \sum_n e^{-in\Theta} \int \mathcal{D}\phi_{(n)}\, e^{-S[\phi]}$$

- How to determine the instanton solution in each class?

Write Euclidean action as

$$\frac{1}{4g^2}\int F^a_{\mu\nu}F^{a\mu\nu} = \underbrace{\frac{1}{8g^2}\int \left(F^a_{\mu\nu} \mp \tilde{F}^a_{\mu\nu}\right)^2}_{\geq 0} \pm \frac{8\pi^2}{g^2} n$$

$$\Rightarrow \quad S \geq \frac{8\pi^2}{g^2}|n|$$

MIN obtained when

$$F^a_{\mu\nu} = \pm \tilde{F}^a_{\mu\nu} \qquad (*)$$

(Fields satisfying $(*)$ are automatically sol. of e.o.m.)

$\Rightarrow$ In the topolgical sector labelled by the integer n

$$S[\phi_{min}] = \frac{8\pi^2}{g^2}|n|$$

$$e^{-V_4 E(\Theta)} = \sum_n e^{-\frac{8\pi^2}{g^2}|n| + in\Theta} \int \mathcal{D}_{(n)}\delta\phi \; e^{-\frac{\delta^2 S}{\delta\phi^2}\big|_{\phi_{min}}\delta\phi} + \dots$$

↑ computable

- As $|n|$ increases, the contribution is suppressed $\Rightarrow$
  $\Rightarrow$ the leading contribution to Θ-dependence is given by $|n| = 1$.

- By doing the computation, one realizes that $E(\Theta)$ has a non trivial Θ-dependence

- Since $E(-\Theta) = E(\Theta)$, any effect of Θ appears in vac. en. at leading order at least through Θ^2 and rapidly disappears for $\theta \ll 1$. Effect of $\theta \neq 0$ most visible in parity-odd observables, like the ELECTRIC DIPOLE MOMENT of the NEUTRON d_n (strongest bound on θ at present).

- El. dipole moment represents the effective coupling between the neutron n and the photon A_μ due to DIFFEREN DISTRIBUTION OF POSITIVE AND NEGATIVE EL. CHARGES inside the neutron. It is formally defined as the couping

$$d_n \, \bar{n} \gamma^{\mu\nu} \gamma_5 n F_{\mu\nu} \quad \leftarrow \text{parity-odd}$$

d_n gets θ-dependence at quantum level (quantum corrections)

$$d_n \sim e|\theta| \frac{m_\pi^2}{M_n^3} \sim 10^{-2} |\theta| \; e \cdot GeV^{-1}$$

$\leftarrow$ from dim. analysi

θ-dependence should disappear if one quark is massless $\rightarrow d_n \propto m_q \propto m_\pi^2$

The most recent experimental bound (2015, Pendlebury et al) is

$$|d_n| < 1.8 \cdot 10^{-12} \; e \cdot GeV^{-1} \Rightarrow \underline{|\theta| \lesssim 10^{-10}} \rightarrow$$

$\rightarrow$ very small $\rightarrow$ STRONG CP PROBLEM :

- A very small dimensionless parameter is naturally related to an approx sym. that is restored when $\theta \to 0$ (quantum correction $\propto \theta$).

Is CP such a sym? No! Because CP already broken by weak interact.