THETA TERM

In YM theory, one has kinetic term

$$-\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} \quad \leftarrow \quad \text{Lorentz \& gauge inv. with at most two derivatives}$$

There is another term that satisfies these conditions:

$$\mathcal{L}_\theta = \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \qquad \text{with} \quad \tilde{F}^{a\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$$

→ This term violates CP. It can be phenomenologically important!
↓

Under CP

$$A^a_0(t,\bar{x}) \mapsto -A^a_0(t,-\bar{x}) \qquad \partial_0 \mapsto \partial_0$$

$$A^a_i(t,\bar{x}) \mapsto A^a_i(t,-\bar{x}) \qquad \partial_i \mapsto -\partial_i$$

Hence CP and P are both violated, then one can have the following terms in the Lagrangian

$$\mathcal{L}_{CPV} = \theta_{QCD} \frac{g_s^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} \;\xleftarrow{}\; SU(3) \; + \theta_2 \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\alpha\beta} W^a_{\mu\nu} W^a_{\alpha\beta} \;\xleftarrow{}\; SU(2)$$

$$+ \theta_1 \frac{g'^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} B_{\mu\nu} B_{\alpha\beta} \;\xleftarrow{}\; U(1)_Y$$

θ - terms

In YM θ parametrizes physically inequivalent theories, despite the fact that $\mathcal{L}_\theta$ is a total derivative

$$\frac{\theta}{64\pi} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \, tr(\mathbb{F}_{\mu\nu} \mathbb{F}_{\rho\sigma})$$

$$= \frac{\theta \epsilon^{\mu\nu\rho\sigma}}{8} tr\left(\partial_\mu \mathbb{A}_\nu \partial_\rho \mathbb{A}_\sigma + 2i \mathbb{A}_\mu \mathbb{A}_\nu \partial_\rho \mathbb{A}_\sigma\right)$$

$$= \frac{\theta}{8} \partial_\rho \left\{ \epsilon^{\mu\nu\rho\sigma} tr\left(\mathbb{A}_\sigma \partial_\mu \mathbb{A}_\nu + \frac{2i}{3} \mathbb{A}_\sigma \mathbb{A}_\mu \mathbb{A}_\nu\right)\right\} \quad (\star)$$

(A) $\Rightarrow$ it does not modify the equations of motion.

However in quantum theory, when π^1(CONFIG. SPACE) is non-trivial, then total derivatives matter.

For non-ab. gauge theories π^1(CONFIG.SP.) $= \mathbb{Z}$.

(*) $\Rightarrow$ it does not contribute to perturb. theory: total derivative is $\propto \sum P_i^{\text{external}}$

However, in presence of fermions charged under G, θ is unphysical because it depends on how the phase of those fermions is chosen.

To understand this effect let us do a REDEFINITION of all fermions:

$$\psi_I \longmapsto e^{i\beta_I \gamma_5} \psi_I$$

I is a FLAVOUR INDEX (For QCD $I=1,\dots,6$)
β_I are real phases

$$\Rightarrow \quad \mathcal{D}\psi\mathcal{D}\bar{\psi} \mapsto e^{-\frac{i}{32\pi^2}\int d^4x\, \epsilon_{\mu\nu\rho\sigma} F_\alpha^{\mu\nu} F_\alpha^{\rho\sigma} \sum_I \beta_I} \mathcal{D}\psi\mathcal{D}\bar{\psi}$$

Axial anomaly

$\rightsquigarrow$ This is equivalent to

$$\theta \mapsto \theta - 2\sum_I \beta_I$$

$\Rightarrow$ Only the combination $\theta - \arg\det(Y_u Y_d)$ is physical (i.e. invariant under field redefinition):

$I = u, d \times 3$

$I = (\alpha, i)$ $i = u, d$ $\alpha, \beta = 1, 2, 3$

$$\sum_{\alpha\beta} Y_u^{\alpha\beta} \bar{q}_L^\alpha \tilde{\Phi} u_R^\beta \mapsto \sum_{\alpha\beta} e^{-i(\beta_\alpha^u + \beta_\beta^u)} Y_u^{\alpha\beta} \bar{q}_L^\alpha \tilde{\Phi} u_R^\beta$$

$\Rightarrow$ phase of $Y_u^{\alpha\beta}$ shifted by $-(\beta_\alpha^u + \beta_\beta^u)$
(i.e. row α multiplied by $e^{-i\beta_\alpha^u}$, column β multiplied by β_β^u)

$$\Rightarrow \det Y_u \to e^{-2i\sum_\alpha \beta_\alpha} \det Y_u$$

Analogously $\det Y_d \to e^{-2i\sum_\alpha \beta_\alpha^d} \det Y_d$

$$\Rightarrow \arg\det Y_u Y_d \mapsto \arg\det Y_u Y_d - 2i \sum_{\substack{\alpha=1,2,3 \\ i=u,d}} \beta_\alpha^i$$

$\to$ the same shift as Θ !

Moreover, thanks to axial anomaly, we can show that Θ and $\Theta + 2\pi$ are physically indistinguishable

In fact, we can always absorb the 2π shift by rotating one ψ_I by $\beta_I = \pi$, that does not affect the Yukawa matrices

$\downarrow$

This can also proved by noticing that $S_\Theta = \Theta \cdot n \quad n \in \mathbb{Z}$

$\Rightarrow e^{iS_\Theta} \to e^{iS_\Theta}$ by taking $\Theta \mapsto \Theta + 2\pi$.

instanton number (topological quantity)

Even if θ-term does not contribute to any order in perturbation theory, nevertheless several SM observables have a non-trivial θ-dependence, such as

- vacuum energy;
- electric dipole moment of the neutron;
- pion mass & proton-neutron mass splitting.

This dependence can be understood qualitatively as the contribution to the SM P.I. of gauge config. with non-vanishing topological charge $Q = \frac{1}{32\pi^2}\int d^4x\, F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$.

Example: VACUUM ENERGY of QCD $E(\theta)$.

- $E(\theta)$ is defined as

$$e^{-V_4 E(\theta)} = \int \mathcal{D}\phi\, e^{-S[\phi] - i\frac{\theta}{32\pi^2}\int F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}}$$

- $E(\theta+2\pi) = E(\theta)$
- $E(-\theta) = E(\theta)$: do field redef. $\phi \to \phi^{CP}$ and use $\mathcal{D}\phi^{CP} = \mathcal{D}\phi$ and $S[\phi^{CP}] = S[\phi]$ for QCD

- $\theta = 0$ is the absolute min. of $E(\theta)$ as

$$e^{-V_4 E(\theta)} = \left| e^{-V_4 E(\theta)} \right| \leq \int \mathcal{D}\phi \left| e^{-S[\phi] - i\frac{\theta}{32\pi^2}\int F\tilde{F}} \right| =$$

$$= \int \mathcal{D}\phi\, e^{-S[\phi]} = e^{-V_4 E(0)}$$

$\Rightarrow \quad E(\theta) \geq E(0)$

$\Rightarrow$ parity sim. $\theta \to -\theta$ is not spont. broken.
(Vafa-Witten theorem)

- Let us extract the form of $E(\theta)$ and show that $E(\theta)\neq 0$.

To do this we need some approximation method to compute the P.I.

⇝ use saddle pt approximation :

- take P.I. in Euclidean formulation as we did above ;
- contribution of integrand mostly given by config. with FINITE ACTION and that MINIMIZE $S[\phi]$;
- expand $S[\phi] = S[\phi_{min}] + \frac{1}{2}\frac{\delta^2 S}{\delta\phi^2}\Big|_{\phi_{min}}\delta\phi^2 + \dots$

it goes out of P.I.

Gaussian integral

⇝ In YM (or QCD) config. A_μ^a that MINIMIZE the Eucl. action and have FINITE ACTION are called INSTANTONS :

- Finite action $\Rightarrow$ $F_{\mu\nu}^a \xrightarrow[|x|\to\infty]{} 0$

$\Rightarrow$ $A_\mu^a \xrightarrow[|x|\to\infty]{}$ PURE GAUGE, i.e. $i\,\partial_\mu U(\hat{x})\,U(\hat{x})^{-1}$ for some $U(\hat{x}) : S^3_\infty \to G$; $\hat{x}\in S^3_\infty$

- Instantons are classified by maps $U : S^3 \to G$; these are classified by HOMOTOPY CLASSES (classes are done by maps that can be CONTINUOUSLY deformed into each other) in $\pi_3(G) = \mathbb{Z}$. The classes are labelled by an INTEGER, that is precisely

$$n = \frac{1}{32\pi^2}\int d^4x\, F_{\mu\nu}^a \tilde{F}^{a\mu\nu} \qquad \text{(INSTANTON NUMBER)}$$

- Maps that have the same inst. numb. have the same Θ-term : $S_\Theta = \Theta n$.

$$e^{-V_4 E(\Theta)} = \sum_n e^{-in\Theta} \int \mathcal{D}\phi_{(n)} \, e^{-S[\phi]}$$

- How to determine the instanton solution in each class? Write Euclidean action as

$$\frac{1}{4g^2}\int F^a_{\mu\nu}F^{a\mu\nu} = \underbrace{\frac{1}{8g^2}\int \left(F^a_{\mu\nu} \mp \tilde{F}^a_{\mu\nu}\right)^2}_{\geq 0} \pm \frac{8\pi^2}{g^2} n$$

$$\Rightarrow \quad S \geq \frac{8\pi^2}{g^2}|n|$$ MIN obtained when

$$F^a_{\mu\nu} = \pm \tilde{F}^a_{\mu\nu} \qquad (*)$$

(Fields satisfying (*) are automatically sol. of e.o.m.)

$\Rightarrow$ In the topolgical sector labelled by the integer n

$$S[\phi_{min}] = \frac{8\pi^2}{g^2}|n|$$

$$e^{-V_4 E(\Theta)} = \sum_n e^{-\frac{8\pi^2}{g^2}|n| + in\Theta} \int \mathcal{D}_{(n)}\delta\phi \, e^{-\frac{\delta^2 S}{\delta\phi^2}\big|_{\phi_{min}}\delta\phi} + \dots$$

computable

- As $|n|$ increases, the contribution is suppressed $\Rightarrow$
  $\Rightarrow$ the leading contribution to Θ-dependence is given by $|n| = 1$.

- By doing the computation, one realizes that $E(\Theta)$ has a non trivial Θ-dependence

- Since $E(-\Theta) = E(\Theta)$, any effect of Θ appears in vac. en. at leading order at least through Θ^2 and rapidly disappears for $\theta \ll 1$. Effect of $\theta \neq 0$ most visible in parity-odd observables, like the ELECTRIC DIPOLE MOMENT of the NEUTRON d_n (strongest bound on θ at present).

- El. dipole moment represents the effective coupling between the neutron n and the photon A_μ due to DIFFEREN DISTRIBUTION OF POSITIVE AND NEGATIVE EL. CHARGES inside the neutron. It is formally defined as the couping

$$d_n \, \bar{n} \gamma^{\mu\nu} \gamma_5 n F_{\mu\nu} \quad \leftarrow \text{parity-odd}$$

d_n gets θ-dependence at quantum level (quantum corrections)

$$d_n \sim e|\theta| \frac{m_\pi^2}{M_n^3} \sim 10^{-2} |\theta| \; e \cdot \mathrm{GeV}^{-1}$$

$M_n^3 \leftarrow$ from dim. analysis

θ-dependence should disappear if one quark is massless $\to d_n \propto m_q \propto m_\pi^2$

The most recent experimental bound (2015, Pendlebury et al) is

$$|d_n| < 1.8 \cdot 10^{-12} \; e \cdot \mathrm{GeV}^{-1} \implies |\theta| \lesssim 10^{-10} \to$$

$\to$ very small $\to$ STRONG CP PROBLEM :

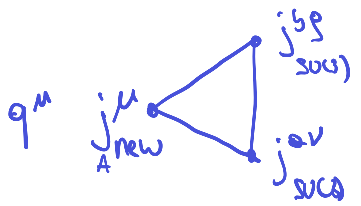
- A very small dimensionless parameter is naturally related to an approx sym. that is restored when $\theta \to 0$ (quantum correction $\propto \theta$).

Is CP such a sym? No! Because CP already broken by weak interact.

- Moreover θ is the sum of two contributions, one of which comes from Yukawa matrices, that are known to have an $\mathcal{O}(1)$ CP violating phase $\rightarrow$
  $\rightarrow$ natural expectation is $\theta \sim \mathcal{O}(1)$.

PECCEI – QUINN SOLUTION

Proposal: there is a new $U(1)$ global symmetry with a mixed anomaly with $SU(3)_c$



$\neq 0$

$\rightarrow$ θ-angle is unphysical if there is a global continuous transf. parametrized by α, that leaves UV theory invariant up to a shift of the gluon top. term $\delta\mathcal{L} \propto \alpha\, F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$

The easiest way to realize it is what proposed by P.Q.:
$S_{UV}[\phi]$ is (classically) invariant under a global $U(1)_{PQ}$ transf., but $U(1)_{PQ}$ is anomalous with mixed triangle diagrams with gluons, so that

$$\mathcal{D}\phi_\alpha = \mathcal{D}\phi\, e^{i\alpha\int F^a_{\mu\nu}\tilde{F}^{a\mu\nu}}$$

First trial would be to check among the already existing $U(1)$ symmetries of the SM. However no one satisfies the requirements:

→ global $U(3)^5$ flav. sym. acting on the Weyl spinors $Q_L, u_R^c, d_R^c, l_L, e_R$ with vanishing mass term, is broken explicitly by $Y_u \neq 0$ $Y_d \neq 0$ to the Baryon and Lepton number : $U(1)_B \times U(1)_L$ that have no anomalous mixed $\triangle$ with $SU(3)_c$

⇒ PQ solution requires new physics beyond SM

- One may add zero-mass fermions that survive in IR to realize the anomalous term. However such fermionic particles are not observed (colored)

- It remains the option that $U(1)_{PQ}$ is SPONT. BROKEN and in the low en. theory its NGB, called QCD AXION $a(x)$, reproduces the anomalous action of the sym in the effective lagrangian.

normalis. of A_μ^a with canonical kinetic term

$$\rightarrow \mathcal{L}_{PQ} = \frac{\alpha_s}{8\pi} \frac{a(x)}{f_a} F^a_{\mu\nu} \tilde{F}^{a\mu\nu} \quad (*) \qquad \alpha_s = \frac{g^2}{4\pi}$$

$$+ \frac{1}{2} \partial_\mu a(x) \partial^\mu a(x)$$

"Axion decay constant" related to scale of sp. sym. breaking

Guessed in the same way as $\mathcal{L}_{\pi\gamma\gamma}$ was guessed.

(*) breakes shift sym(*) ⇒ axion mass expected $m_a \sim f_a^{-1}$ by quantum corrections

$\Rightarrow m \propto \Lambda^4_{QCD}/f_a^2$

(*) Shift sym forbids mass term for axion a

At energies below scale of PQ sym. breaking, the SM + axion Lagrangian is

MODEL INDEP. term

$$\mathcal{L} = \mathcal{L}_{SM}^{\theta=0} + \theta \frac{\alpha_s}{8\pi} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \frac{1}{2}\partial_\mu a\, \partial^\mu a + \frac{\alpha_s}{8\pi}\frac{a}{f_a} F^a_{\mu\nu}\tilde{F}^{a\mu\nu} + \dots \qquad (*)$$

PQ sym now realized as SHIFT sym
$a \to a + \alpha f_a$

model dependent pieces

terms that do not break shift sym and $\propto (f_a^{-1})^k$

"..." include terms like

$$\frac{g^0_{a\gamma\gamma}}{4} a F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{\partial_\mu a}{2 f_a}\sum_i c_i^0 \bar{f}_i \gamma^\mu \gamma_5 f_i \qquad (g^0_{a\gamma\gamma} \propto f_a^{-1})$$

either a quark or a lepton

there could be non-derivative coupling of axion to quark mass matrix $\bar{q}\, e^{i c_q \gamma_5 a/f_a} M_q\, q$

but it can be rotated away by redefining $q \to e^{-(c_q/2) i\gamma_5 a/f_a} q$
that would just shift the $aF\tilde{F}$ coupling and $\partial_\mu a\, \bar{q}\gamma^\mu\gamma_5 q$.

$\rightarrow$ f_a is defined by $(*)$ in the frame where $c_q = 0$.

Now we can use shift sym to rotate away the θ-term ($\alpha = -\theta$). All the rest of $\mathcal{L}$ is left invariant.

$\Rightarrow$ in presence of AXION - GLUON coupling, the θ-term is not physical and all observables are indep. of θ.
(results are equiv. to $\theta=0$ $\leadsto$ resolution of Strong CP problem)

$g_{a\gamma\gamma}$ and c_q coupling are important because they give potential processes where axion could be detected.
↳ these coup. are prop. to f_a^{-1} powers ⇝ way to constrain f_a value.

Astrophys. observ. based on neutrino emission from SUPERNOVA 1987 A give

$$f_a \gtrsim 10^8 \text{ GeV}$$

← very weak interaction with visible matter

$\Rightarrow m_a \lesssim 1$ eV (very small)

An upper bound comes from black hole superradiance

$$f_a \lesssim 2 \cdot 10^{17} \text{ GeV}$$

Depending on cosmological scenario f_a gets more stringent upper bound from Dark Matter production.

Since m_a is very small, a will survive in IR below QCD scale $\Rightarrow$ we can define an effective action with $\underbrace{\text{(light) MESONS}}_{\text{PIONS}}$ and a.

To do this we will use the following trick: rotate q's s.t. axion-gluon coup. disappears; the price to pay is a mass term dressed with axion field a.
We know how M_q appear in low en. Lagrangian.

$$\mathcal{L}_\pi = f_\pi^2 \, \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger - B_0 f_\pi^2 \, \mathrm{Tr}(U M_q^\dagger + M_q U^\dagger)$$

Now substitute

$$M_q \to e^{i c_q \gamma_5 a/2f_a} M_q \, e^{i c_q \gamma_5 a/2f_a}$$

That reduces to expression above when c_q is prop to $\mathbb{1}$.

⇒ We then obtain a scalar potential depending on pions and axion. Pions have mass bigger than $m_a \lesssim 1\,eV$

⇒ we can integrate pions out and obtain the axion potential

$$V(a) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_d + m_u)^2} \sin^2\left(\frac{a}{2f_a}\right)}$$

→ Even and periodic potential

→ MIN. is at $a=0$ ⇝ CP is not spont. broken

- Including next to leading corrections:

$$V(a) \to V(a) \cdot \Big(1 + \underbrace{\mathcal{O}\left(\frac{m_u}{m_s}\right)}_{\sim 1/40} + \underbrace{\text{uncertainty on } m_u, m_d}_{\sim 1/40}\Big)$$

from including also quench s.

- One can derive an expression for axion mass

$$m_a^2 = \frac{m_u m_d}{(m_u + m_d)^2} \frac{m_\pi^2 f_\pi^2}{f_a^2} \longrightarrow m_a \sim 5{,}7\ \mu eV \left(\frac{10^{12}\,GeV}{f_a}\right)$$

- There is also a $\frac{\lambda}{4!} a^4$ self-interaction with

$$\lambda = -\frac{m_u^3 + m_d^3}{(m_u + m_d)^3} \frac{m_a^2}{f_a^2} \sim 10^{-53} \left(\frac{10^{12}\,GeV}{f_a}\right)$$

↳ very weak (unless there are many axions, that generate an attractive force even competing with gravity.