Nov 16

$E \qquad E', \qquad \sigma(E, E')$

Th E loc convex. Then if C is a convex subset, C is strongly closed if and only if it is closed for $\sigma(E, E')$ top.

Corollary E locally convex and $\phi : E \to \mathbb{R}$ convex. Then ϕ is lower semicontinuous in the strong topology if and only if it is lower semicontinuous in the $\sigma(E, E')$ top

Pf Recall that ϕ is lower semicontinuous if for any $a \in \mathbb{R}$

$$C_a = \phi^{-1}((-\infty, a]) \text{ is closed}$$

Since ϕ convex, C_a is convex for any a.

$x_0, x_1 \in C_a, \quad \phi(x_0) \leq a, \quad \phi(x_1) \leq a$

$x_t = (1-t)\, x_0 + t x_1$

$$\phi(x_t) \leq (1-t)\,\phi(x_0) + t\,\phi(x_1)$$

$$\leq (1-t)\,\alpha + t\,\alpha = \alpha$$

$\phi(x_t) \leq \alpha \implies C_\alpha$ is convex

C_α is closed strongly $\iff C_\alpha$ is closed for $\sigma(E, E')$

<u>Theorem</u> $T : E \to F$ T linear, E and F B spaces

1) T is ~~strongly~~ continuous iff

2) $T : (E, \sigma(E, E')) \longrightarrow (F, \sigma(F, F'))$ is continuous

<u>Pf</u> 1) $\Rightarrow$ 2) $T : X \longrightarrow E$ weak $\downarrow y'$ $\mathbb{R}$

In order to prove 2) it is enough ~~to~~ show that $y' \circ T \in E'$ $\forall\, y' \in F'$

Indeed $y' \circ T$ defines an element in E'

and $y' \circ T : (E, \sigma(E, E')) \longrightarrow \mathbb{R}$

is continuous $\implies$ 2 is true

$2 \Rightarrow 1$

$$T : (E, \sigma(E, E')) \longrightarrow (F, \sigma(F, F'))$$

$$G(T) \subseteq E \times F$$

$\sigma(E \times F, (E \times F)')$ is the product topology of $(E, \sigma(E, E')) \times (F, \sigma(F, F'))$

$G(T)$ is a vector space is closed in it

so is closed in the $\sigma(E \times F, (E \times F)')$ topology $\Longrightarrow$ $G(T)$ is strongly closed in $E \times F$

$E \xrightarrow{T} F$ linear $G(T)$ closed

$$\Longrightarrow \quad T \in \mathcal{L}(E, F)$$

$\ell^p(\mathbb{N}) \qquad 1 < p < +\infty$

Exercise in $\ell^1(\mathbb{N}) = \{ f : \mathbb{N} \to \mathbb{R} : \sum_{j=1}^{\infty} |f(j)| < +\infty \}$

A sequence $\{f_n\}$ converges strongly if and only if it converges weakly

$$\sigma(\ell^1(\mathbb{N}), \ell^\infty(\mathbb{N}))$$

Yet the topologies are different.

$f \in \overline{L^p(\mathbb{R}^d)}$

$$Mf(x) = \sup_{R>0} \frac{1}{|D(x,R)|} \int_{D(x,R)} |f(y)|\, dy$$

$$M(f+g) \le M(f) + M(g)$$

$$M : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$$

$1 < p \le +\infty$ $\qquad \exists \; C_p = \frac{1}{p-1} \cdots$

$$\|Mf\|_{L^p} \le C_p \|f\|_{L^p} \qquad \forall \, f \in L^p(\mathbb{R}^d)$$

$$\hat{f}(\xi) \longrightarrow \operatorname{sign} \xi \; \hat{f}(\xi)$$

$$L^p(\mathbb{R}) \longrightarrow L^p(\mathbb{R}) \qquad 1 < p < +\infty$$

$$f \longrightarrow Hf(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{1}{x-y} f(y)\, dy$$

$\sigma(E', E)$ topology

E B-space $\qquad E'$ is a B-space

$\sigma(E', E'')$

Def Give E', the weak $\sigma(E', E)$ topology has a subbasis of seminorms the family

$$\{ | \langle \ , x \rangle_{E' \times E} | \}_{x \in E}$$

Remark A basis of neigh of $0 \in E'$ is of the form

$$V = \{ f : |f(x_j)| < \varepsilon, \ x_1, \dots, x_n \in E \}$$

$\varepsilon > 0$

Lemma E' is Hausdorff for the $\sigma(E', E)$.

Pf. Given $f_0 \neq f_1$ in E', the exists $x \in E$ and $\alpha \in \mathbb{R}$ s.t. $f_0(x) < \alpha < f_1(x)$

$$\{ f \in E' : f(x) < \alpha \} \ni f_0$$

$$\{ f \in E' : f(x) > \alpha \} \ni f_1$$

If $f_n \longrightarrow f$ in E' in the $\sigma(E', E)$

we will write $f_n \xrightarrow{*} f$

$$f_n \Rrightarrow f$$

<u>Lemma</u> Given $f_n \in E'$, $f \in E'$

1) $f_n \xrightarrow{*} f \iff f_n(x) \to f(x) \ \forall x \in E$

2) $f_n \to f$ strongly in E' $\Rightarrow$ $f_n \xrightarrow{*} f$

3) $f_n \xrightarrow{*} f$ then $\sup \{ \|f_n\|_{E'} \} < +\infty$

and $\|f\|_{E'} \leq \liminf \|f_n\|_{E'}$

4) $f_n \xrightarrow{*} f$ and $x_n \to x$ strongly in E

then $f_n(x_n) \to f(x)$.

<u>Example</u> $c_0(\mathbb{N}) = \{ \{x_n\}_{n\in\mathbb{N}} : \lim_{n\to+\infty} x_n = 0 \}$

$$\subsetneq \ell^\infty(\mathbb{N})$$

c_0 is strongly closed in ℓ^∞ $\quad \left(C_0^0(\mathbb{R}^d) \subsetneq L^\infty(\mathbb{R}^d) \right)$

but is not closed for $\sigma(\ell^\infty, \ell^1)$ top.

$$X_n = (\underbrace{1, \dots, 1}_{n}, 0, 0 \dots\dots 0) \quad \text{in } c_0(\mathbb{N})$$

$$\xi \in \ell^1(\mathbb{N})$$

$$\langle X_n, \xi \rangle_{\ell^\infty(\mathbb{N}) \times \ell^1(\mathbb{N})} =$$

$$= \xi_1 + \dots + \xi_n \xrightarrow{n \to +\infty} \sum_{n=1}^{\infty} \xi_n = \langle X, \xi \rangle_{\ell^\infty \times \ell^1}$$

$$\langle \{x_n\}, \{y_n\} \rangle = \sum_{n=1}^{\infty} x_n y_n$$

where $X = (1, 1, \dots\dots 1, \dots\dots)$

$$\underset{\substack{\uparrow \\ c_0(\mathbb{N})}}{X_n} \xrightarrow{\not\,} X \in \ell^\infty(\mathbb{N}) \setminus c_0(\mathbb{N})$$

Exercise E B-space, $\dim E = +\infty$, then $(E', \sigma(E', E))$

is not metrizable, not even if E is separable

that is, when $\exists$ $X \subset E$ X countable

$\overline{X} = E$.

$$\{ |\langle\ , x\rangle_{E^* \times E}| \}_{x \in E}$$

$$\{ |\langle\ , x\rangle_{E' \times E}| \}_{x \in X}$$

$D(0,1)$
E'

Ex $f \in L^\infty(\mathbb{R}^d)$ supp f compact

Then if $\lambda_n \xrightarrow{n \to +\infty} +\infty$ $\{\lambda_n\}$ in $\mathbb{R}^+$

$$f_{\lambda_n}(x) = f(\lambda_n x)$$

$$f_\lambda(x) = \lambda^{\frac{d}{p}} f(\lambda x)$$

check $f_n \xrightarrow{*} 0$

$$\lambda_n \xrightarrow{n \to +\infty} 0^+$$

Proposition Given $\phi : E' \to \mathbb{R}$ linear and continuous for $\sigma(E', E)$, then $\exists\, x \in E$

$$\phi(f) = f(x)$$

Lemma X a vector space and $f, f_1, \dots, f_n$ linear maps $X \to \mathbb{R}$. Suppose that

$$f_1(x) = \dots = f_n(x) = 0 \implies f(x) = 0$$

Then $f = \lambda_1 f_1 + \dots + \lambda_n f_n$ for

$\lambda_1, \dots, \lambda_n \in \mathbb{R}$

<u>Pf</u> $F : X \to \mathbb{R}^{n+1}$

$$F(x) = (f(x), f_1(x), \dots, f_n(x))$$

$$R(F) \subseteq \mathbb{R}^{n+1}$$

$$A = (1, 0, \dots, 0) \notin R(F)$$

$\exists$ linear map $\mathbb{R}^{n+1} \to \mathbb{R}$ which separates

$A \qquad R(F)$

$\Lambda \in \mathbb{R}^{n+1}$

$\Lambda = (\lambda, \lambda_1, \dots, \lambda_n)$

$$A \cdot \Lambda = \lambda < \alpha < \underbrace{F(x) \cdot \Lambda = \lambda f(x) + \lambda_1 f_1(x) + \dots + \lambda_n f_n(x)}_{\equiv 0}$$

$\cancel{\forall x \in R(F)}$ $\qquad \forall x \in X$

$$\lambda < \alpha < 0$$

$= 0 \quad \forall x \in X$

$$f(x) = -\frac{\lambda_1}{\lambda} f_1(x) + \dots + \frac{\lambda_n}{\lambda} f_n(x) \quad \forall x \in X$$

$\phi : E' \to \mathbb{R}$ continuous for $\sigma(E', E)$

$\Rightarrow \quad \phi(V) \subset (-1, 1)$ $\quad$ V open nbh of 0

$$V = \{ f : |f(x_j)| < \varepsilon, \ x_1, \dots, x_n \in E \}$$

$|\phi(f)| < 1$ ~~p~~ if $\quad |f(x_j)| < \varepsilon, \ x_1, \dots, x_n \in E$

In particular if $f \in E'$ s.t.

$$f(x_j) = 0 \quad \forall \ j = 1, \dots, n$$

$$tf(x_j) = 0 \implies \phi(tf) = 0$$

$$\phi(f) = 0$$

$\phi \qquad E' \xrightarrow{\phi} \mathbb{R}$

x_1

$\vdots$

x_n

$$\phi = \lambda_1 x_1 + \dots + \lambda_n x_n$$