16 Novembre

$(c)' = 0$, $(e^x)' = e^x$

Lemma $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$

Dim

$$\frac{\sin(x) - \sin(x_0)}{x - x_0} = \qquad x = h + x_0$$

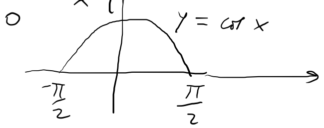
$$= \frac{\sin(h+x_0) - \sin(x_0)}{h} = \frac{\sin(h)\cos(x_0) + \cos(h)\sin(x_0) - \sin(x_0)}{h}$$

$$= \frac{\sin(h)}{h}\cos(x_0) + \sin(x_0)\frac{\cos(h) - 1}{h}$$

$$\lim_{x \to x_0} \frac{\sin(x) - \sin(x_0)}{x - x_0} = \cos(x_0) \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}_{1} + \sin(x_0) \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{0}$$

$$= \cos(x_0)$$

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \iff \lim_{x \to 0} \left| \frac{1 - \cos x}{x} \right| = 0$$



$$|x| < \frac{\pi}{2} \Rightarrow \cos(x) > 0$$

$0 < |x| < \frac{\pi}{2}$

$$0 \le \left| \frac{1 - \cos x}{x} \right| = \frac{1}{|x|} \frac{1 - \cos x}{1 + \cos x} (1 + \cos x) = \frac{1}{|x|} \frac{1 - \cos^2 x}{1 + \cos x}$$

$$= \frac{\sin^2 x}{|x|} \frac{1}{1 + \cos x} < \frac{\sin^2 x}{|x|} = \left| \frac{\sin^2 x}{x} \right| = \left| \frac{\sin x}{x} \sin x \right|$$

$$= \left| \frac{\sin x}{x} \right| |\sin x| \xrightarrow{x \to 0} 1 \cdot 0 = 0$$

$\left|\frac{\sin x}{x}\right| \downarrow_{x \to 0} 1$, $|\sin x| \downarrow 0$

$$\lim_{x \to 0} \left| \frac{\sin x}{x} \right| = \lim_{y \to 1} |y| = |1| = 1$$

$y = \frac{\sin x}{x}$

$$\lim_{x \to 0} |\sin x| = \lim_{y \to 0} |y| = |0| = 0$$

$y = \sin x$

$\sin'(x) = \cos x \quad \forall x \in \mathbb{R}$

$a \in \mathbb{R}, \quad x^a : \mathbb{R}_+ \to \mathbb{R}$

$(x^a)' = a\, x^{a-1}$

Dim

$$\lim_{x \to x_0} \frac{x^a - x_0^a}{x - x_0} = \qquad x = h + x_0$$

$$= \lim_{h \to 0} \frac{(h+x_0)^a - x_0^a}{h} = \lim_{h \to 0} \frac{\left(x_0\left(1+\frac{h}{x_0}\right)\right)^a - x_0^a}{h}$$

$$= \lim_{h \to 0} \frac{x_0^a \left(1+\frac{h}{x_0}\right)^a - x_0^a}{h} = x_0^a \lim_{h \to 0} \frac{\left(1+\frac{h}{x_0}\right)^a - 1}{h}$$

Ora ricordiamo $\lim_{y \to 0} \frac{(1+y)^a - 1}{y} = a$

$y = \frac{h}{x_0}, \quad h = y x_0$

$$= x_0^a \lim_{y \to 0} \frac{(1+y)^a - 1}{x_0 y} = x_0^{a-1} \lim_{y \to 0} \frac{(1+y)^a - 1}{y}$$

$$= a\, x_0^{a-1}$$

Abbiamo dimostrato $(x^a)' = a\, x^{a-1} \quad \forall\, x > 0$ e $\forall a \in \mathbb{R}$

Lemma $(\lg x)' = \frac{1}{x} \quad \forall \quad x > 0$

Dim $\lim_{x \to x_0} \frac{\lg x - \lg x_0}{x - x_0} \qquad x = h + x_0$

$$\frac{\lg x - \lg x_0}{x - x_0} = \frac{\lg(h + x_0) - \lg(x_0)}{h} = \frac{\lg\left(x_0\left(1 + \frac{h}{x_0}\right)\right) - \lg(x_0)}{h}$$

$$= \frac{\cancel{\lg(x_0)} + \lg\left(1 + \frac{h}{x_0}\right) - \cancel{\lg(x_0)}}{h} = \frac{\lg\left(1 + \frac{h}{x_0}\right)}{h} =$$

$$\lim_{y \to 0} \frac{\lg(1+y)}{y} = 1 \qquad y = \frac{h}{x_0}$$

$$= \frac{\lg(1+y)}{x_0 y} = x_0^{-1} \frac{\lg(1+y)}{y}$$

$$\lim_{h \to 0} \frac{\lg(h + x_0) - \lg(x_0)}{h} = \lim_{h \to 0} \frac{\lg\left(1 + \frac{h}{x_0}\right)}{h} =$$

$$= \lim_{y \to 0} \frac{1}{x_0} \frac{\lg(1+y)}{y} = \frac{1}{x_0} \underbrace{\lim_{y \to 0} \frac{\lg(1+y)}{y}}_{=1} = x_0^{-1}$$

Cioè $(\lg x)' = x^{-1}$

Lemma Sia $f: I \to \mathbb{R}$, $x_0 \in I$. Se esiste $f'(x_0)$

allora f è continua in x_0

Dim Per ipotesi esiste ed è finito $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0)$

f continua in $x_0 \iff \lim_{x \to x_0} f(x) = f(x_0) \iff$

$$\lim_{x \to x_0} \left(f(x) - f(x_0) \right) = 0$$

$$\lim_{x \to x_0} \left(f(x) - f(x_0) \right) = \lim_{x \to x_0} \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{f'(x_0)} \underbrace{(x-x_0)}_{0} = f'(x_0)\, 0$$

$$= 0$$

Teor Siano $f, g : I \to \mathbb{R}$ $x_0 \in I$. Supponiamo esistano $f'(x_0)$ e $g'(x_0)$. Valgono

1) $(f+g)'(x_0) = f'(x_0) + g'(x_0)$

2) $(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$

3) Se $g(x) \neq 0$ $\forall$ $\left(\dfrac{1}{g(x)}\right)'(x_0) = -\dfrac{g'(x_0)}{g^2(x_0)}$

4) $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{g^2(x_0)}$ $(4 \Rightarrow 3)$

Osservazioni $(4) \Rightarrow (3)$. Ma $(2) \,\&\, (3) \Rightarrow (4)$

$$\left(\frac{f}{g}\right)' = \left(f \, \frac{1}{g}\right)'(x) \overset{(2)}{=} f'(x)\, \frac{1}{g(x)} + f(x) \left(\frac{1}{g(x)}\right)'$$

$$\overset{(3)}{=} \frac{f'(x)}{g(x)} + f(x)\left(-\frac{g'(x)}{g^2(x)}\right) = \frac{f'(x)}{g(x)} - f(x)\,\frac{g'(x)}{g^2(x)}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g^2(x)}$$

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

$$\frac{(f+g)(x) - (f+g)(x_0)}{x - x_0} = \frac{(f(x)+g(x)) - (f(x_0)+g(x_0))}{x - x_0} =$$

$$= \frac{(f(x)-f(x_0)) + (g(x)-g(x_0))}{x - x_0} = \frac{f(x)-f(x_0)}{x - x_0} + \frac{g(x)-g(x_0)}{x - x_0}$$

$$\frac{f(x)-f(x_0)}{x - x_0} \xrightarrow{x \to x_0} f'(x_0) \qquad \frac{g(x)-g(x_0)}{x - x_0} \xrightarrow{x \to x_0} g'(x_0)$$

$$\xrightarrow{x \to x_0} f'(x_0) + g'(x_0)$$

$$\frac{(fg)(x)-(fg)(x_0)}{x-x_0} = \frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} =$$

$$= \frac{\left(f(x)g(x)-f(x_0)g(x)\right)+\left(f(x_0)g(x)-f(x_0)g(x_0)\right)}{x-x_0} =$$

$$= \underbrace{\frac{f(x)-f(x_0)}{x-x_0}}_{\xrightarrow{x\to x_0}\ f'(x_0)} \underbrace{g(x)}_{\xrightarrow{x\to x_0}\ g(x_0)} + \underbrace{f(x_0)}_{\to f(x_0)} \underbrace{\frac{g(x)-g(x_0)}{x-x_0}}_{\xrightarrow{x\to x_0}\ g'(x_0)}$$

$$\lim_{x\to x_0} \frac{(fg)(x)-(fg)(x_0)}{x-x_0} = f'(x_0)g(x_0)+f(x_0)g'(x_0)$$

<u>Osservazione</u> $\quad (fg)'(x) = f'(x)g(x)+f(x)g'(x)$

se $g \equiv c \implies (cf)' = c f'$

$$\frac{d}{dx} cf = c \frac{d}{dx} f$$

Combinando con la regola della somma, per $\lambda, \mu \in \mathbb{R}$

$$\frac{d}{dx}\left(\lambda f(x)+\mu g(x)\right) = \frac{d}{dx}\left(\lambda f(x)\right) + \frac{d}{dx}\left(\mu g(x)\right)$$

$$= \lambda \frac{d}{dx} f(x) + \mu \frac{d}{dx} g(x)$$

$$(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x$$

?

$$\frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

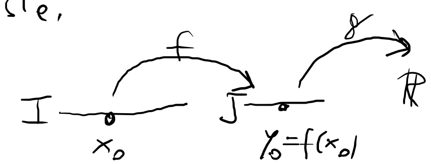
$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} =$$

$$= \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Teor I, J due intervalli, $f : I \to J$, $g : J \to \mathbb{R}$

$x_0 \in I$, $y_0 \in J$, $f'(x_0)$ esiste, $g'(y_0)$ esiste.

Considerato $y_0 = f(x_0)$ $g(f(x)) : I \to \mathbb{R}$, si



$$\big(g(f(x))\big)'(x_0) = f'(x_0)\, g'(y_0)$$

Osservazione $y = f(x)$

$$\frac{d\,g(y)}{dx} = \frac{d\,g(y)}{d\cancel{x}}\;\frac{dy}{dy} = \frac{d\,g(y)}{dy}\;\frac{dy}{dx}$$

Corollario $\left(\frac{1}{g}\right)'(x_0) = -\,\frac{g'(x_0)}{g^2(x_0)}$

Dim $\frac{1}{g(x)} = F(g(x))$ $\qquad F(y) = \frac{1}{y} = y^{-1}$

$F'(y) = -1\; y^{-2} = -\,y^{-2}$

Se $y_0 = g(x_0)$

$$\big(F(g(x))\big)'(x_0) = F'(y_0)\; g'(x_0) = -\frac{1}{y_0^2}\, g'(x_0) = -\frac{1}{g^2(x_0)}\, g'(x_0)$$

$\exp y = e^y$

Esempio $\left(e^{f(x)}\right)' = \exp'(f(x))\; f'(x) = \exp(f(x))\; f'(x)$

$= e^{f(x)}\; f'(x)$

$$(b^x)' = \left(e^{\lg b^x}\right)' = \left(e^{x \lg b}\right)' = e^{x\lg b}\;(x \lg b)' =$$

$$= b^x\; \lg b\; (x)' = b^x \lg b$$

Esempio (Modello di Malthus)

E' un modello di popolazione dove la variazione della popolazione ~~nell' unità di~~ nel tempo dipende dalla popolazione iniziale, dalla lunghezza dell' intervallo di tempo e da una costante specifica k.

$t \to n(t)$

$$n(t) - n(t_0) = k \, n(t_0) \, (t - t_0)$$

$$\Delta n = k \, n \, \Delta t$$

$$\frac{\Delta n}{\Delta t} = k \, n$$

$$\frac{dn}{dt} = k \, n \iff n(t) = C_0 e^{tk}$$

$$n(0) = C_0$$

$$\boxed{n(t) = n_0 e^{tk}}$$

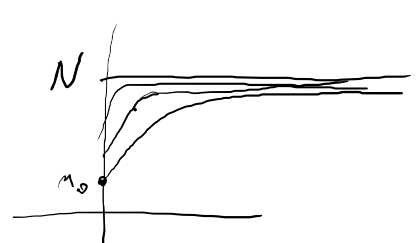
N

Equazione logistica

$$\boxed{\Delta n = n \, k \, \Delta t \left(1 - \frac{n}{N}\right)}$$

$$\frac{dn}{dt} = n \, k \left(1 - \frac{n}{N}\right)$$

$$n(t) = \frac{N n_0 \, e^{kt}}{N + n_0 (e^{kt} - 1)}$$



Teo $f: I \to J$, $g: J \to \mathbb{R}$ $x_0 \in I$, $y_0 = f(x_0) \in J$

$\exists$ $f'(x_0)$ e $g'(y_0)$. Allora

$$\Big(g\big(f(x)\big)\Big)'(x_0) = g'(y_0)\, f'(x_0)$$

Dim In linea di massima, verrebbe da scrivere. Se f e' strett. monotona

$$\frac{g(f(x_0+h)) - g(f(x_0))}{h} = \frac{g(f(x_0+h)) - g(f(x_0))}{f(x_0+h) - f(x_0)} \quad \frac{f(x_0+h) - f(x_0)}{h}$$

$$\lim_{h\to 0} \frac{g(f(x_0+h)) - g(f(x_0))}{h} = \lim_{h\to 0} \frac{g(f(x_0+h)) - g(y_0)}{f(x_0+h) - y_0} \underbrace{\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}_{f'(x_0)}$$

$y = f(x_0+h) \xrightarrow{h\to 0} f(x_0) = y_0$

$$= f'(x_0) \lim_{y\to y_0} \frac{g(y) - g(y_0)}{y - y_0} = f'(x_0)\, g'(y_0)$$

Per dimostrare in modo rigoroso, e più generale, si procede come segue.

In J definiamo una nuova funzione

$$G(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} & \text{se } y \neq y_0 \\ g'(y_0) & \text{se } y = y_0 \end{cases}$$

Notiamo che $\lim_{y\to y_0} G(y) = G(y_0) = g'(y_0)$ ed osserviamo che risulta

$$* \quad \frac{g(f(x_0+h)) - g(f(x_0))}{h} = G(f(x_0+h)) \quad \frac{f(x_0+h) - f(x_0)}{h}$$

Se $f(x_0+h) \neq f(x_0)$, allora $*$ diviene

$$\frac{g(f(x_0+h)) - g(f(x_0))}{h} = \frac{g(f(x_0+h)) - g(f(x_0))}{\cancel{f(x_0+h) - f(x_0)}} \quad \frac{\cancel{f(x_0+h) - f(x_0)}}{h}$$

Se invece $f(x_0+h) = f(x_0)$ $*$ continua ad essere vera, perché

$$\frac{g(\overbrace{f(x_0+h)}) - g(\overbrace{f(x_0)})}{h} = \frac{g(f(x_0)) - g(f(x_0))}{h} = 0 \quad \Big/ \quad G(\overbrace{f(x_0+h)}^{f(x_0)}) \quad \frac{\overbrace{f(x_0+h) - f(x_0)}^{0}}{h} = 0$$

$$\lim_{h\to 0} \frac{g(f(x_0+h)) - g(f(x_0))}{h} = \lim_{h\to 0} G(f(x_0+h)) \underbrace{\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}}_{= f'(x_0)}$$

$y = f(x_0+h) \xrightarrow{h\to 0} f(x_0) = y_0$

$$= f'(x_0) \lim_{y\to y_0} G(y) = f'(x_0)\, G(y_0) = f'(x_0)\, g'(y_0).$$