Δ_{gh}

$\Delta_1 = -i\,\partial^\mu A_\mu - i A_\mu \partial^\mu$

$$-\frac{1}{2}\,\mathrm{Tr}\left((-\partial^2)^{-1}\Delta_1(-\partial^2)^{-1}\Delta_1\right) = -\frac{1}{2}\int d^d y\,\langle y|\,\mathrm{tr}\left((-\partial^2)^{-1}\Delta_1(-\partial^2)^{-1}\Delta_1\,|y\rangle\right) =$$

?

$$= +\frac{1}{2}\int d^d y\,\langle y|\Big[\mathrm{tr}\left((-\partial^2)^{-1}\partial^\mu \overset{\int d^dx|x\rangle\langle x|}{\vee} A_\mu^a t^a_{Ad}(-\partial^2)^{-1}\partial^\nu A_\nu^b t^b_{Ad}\right) +$$

$$+\,\mathrm{tr}\left((-\partial^2)^{-1}A_\mu^a t^a_{Ad} \overset{\int d^dx|x\rangle\langle x|}{\vee}\partial^\mu(-\partial^2)^{-1}\partial^\nu A_\nu^b t^b_{Ad}\right) +$$

$$+\,\mathrm{tr}\left((-\partial^2)^{-1}\partial^\mu \overset{\int d^dx\,|x\rangle\langle x|}{\vee} A_\mu^a t^a_{Ad}(-\partial^2)^{-1}A_\nu^b\partial^\nu t^b_{Ad}\right) +$$

ciclicità tracce

$$+\,\mathrm{tr}\left((-\partial^2)^{-1}A_\mu^a t^a_{Ad} \overset{\int d^dx|x\rangle\langle x|}{\vee}\partial^\mu(-\partial^2)^{-1}A_\nu^b T^b\,\partial^\nu\right)\Big]|y\rangle$$

Consideriamo il primo termine

$\langle y|k\rangle = e^{iky}$

$$1^{st}\ \text{term} = \frac{1}{2}\int d^dx\,d^dy\ \mathrm{tr}\left[A_\mu^a(x)\,t^a_{Ad}\,A_\nu^b(y)\,t^b_{Ad}\right].$$

k' - completezza

$$\langle y|(-\partial^2)^{-1}\partial^\mu \overset{\int \frac{d^dk}{(2\pi)^d}|k\rangle\langle k|}{\vee}|x\rangle\,\langle x|(-\partial^2)^{-1}\partial^\nu\overset{}{\vee}|y\rangle$$

$\langle y|k\rangle = e^{iky}$

$$\frac{1}{-(ik)^2}\,ik^\mu\,e^{ik(y-x)} \qquad \frac{1}{-(ik')^2}\,ik'^\nu\,e^{-ik'(y-x)}$$

$$-\frac{k^\mu}{k^2}\frac{k'^\nu}{k'^2}\,e^{i(k-k')(y-x)}$$

2^{nd} term $= \quad \ldots$ ← stessa prima riga del 1^{st} term

$$\langle y|(-\partial^2)^{-1}|x\rangle\cdot\langle x|\,\partial^\mu(-\partial^2)^{-1}\partial^\nu\,|y\rangle$$

$$+\frac{1}{k^2}\cdot\frac{(-)k'^\mu k'^\nu}{k'^2}\,e^{i(k-k')(y-x)}$$

3^{rd} term $\qquad - \frac{k^\mu k^\nu}{k^2} \cdot \frac{1}{k^2} \quad e^{i(k-k')(y-x)}$

4^{th} term $\qquad - \frac{k^\nu}{k^2} \quad \frac{k'^\mu}{k^2} \quad e^{i(k-k')(y-x)}$

$$\Downarrow$$

$$1^{st} + 2^{nd} + 3^{rd} + 4^{th} = - \frac{e^{i(k-k')(y-x)}}{k^2 k'^2} (k+k')^\mu (k+k')^\nu$$

Quindi:

$$-\frac{1}{2} \mathrm{Tr}\left((-\partial^2)^{-1} \Delta_1 (-\partial^2)^{-1} \Delta_1 \right) =$$

$$= -\frac{1}{2} \int d^d x \, d^d y \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \, e^{i(k-k')(y-x)} \frac{(k+k')^\mu (k+k')^\nu}{k^2 k'^2} \mathrm{tr}\left(A^a_\mu(x) t^a_{Ad} A^b_\nu(y) t^b_{Ad} \right)$$

$$\tilde{A}(k) = \int d^d x \, e^{ikx} A(x) \rightarrow A(x) = \int \frac{d^d p}{(2\pi)^d} e^{-ipx} \tilde{A}(p)$$

$$A(y) = \int \frac{d^d q}{(2\pi)^d} e^{-iqy} \tilde{A}(q)$$

$$\downarrow$$

$$-\frac{1}{2}\int d^dx\, d^dy\, \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\; \underbrace{e^{i(k-k')y}\; e^{-iqy}}_{\delta(k-k'-q)}\; \underbrace{e^{-i(k-k')x}\; e^{-ipx}}_{\delta(k-k'+p)}\;\cdot$$

$$\cdot\frac{(k+k')^\mu (k+k')^\nu}{k^2\, k'^2}\; \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(q)\; \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right)$$

$$\begin{cases} k-k'-q=0 \\ k-k'+p=0 \end{cases} \Rightarrow \begin{cases} q=-p \\ k'=p+k \end{cases}$$

$$= -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) \tilde{A}^a_\mu(p)\, \tilde{A}^b_\nu(-p) \int \frac{d^dk}{(2\pi)^d}\, \frac{(p+2k)^\mu (p+2k)^\nu}{k^2\,(k+p)^2}$$

Riscriviamo anche il primo termine $\int d^dx\, A^{\mu a}(x)\, A^b_\mu(x)\, \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$ nello sp. dei momenti:

$$\mathrm{Tr}\left((-\partial^2)\Delta_2\right) = \int d^dx \int \frac{d^dp}{(2\pi)^d}\frac{d^dq}{(2\pi)^d}\frac{d^dk}{(2\pi)^d}\, \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) \tilde{A}^{\mu a}(p)\, \tilde{A}^b_\mu(q)\, \underbrace{\frac{1}{k^2}\, e^{-ipx}\, e^{-iqx}}_{\delta(p+q)}$$

$$= \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) \tilde{A}_\mu(p)\, \tilde{A}_\nu(-p) \int \frac{d^dk}{(2\pi)^d}\, \frac{\delta^{\mu\nu}}{k^2}$$

siamo su sp. Euclideo

$$\mathrm{Tr}\log\Delta_{gh} = \int \frac{d^dp}{(2\pi)^d}\, \mathrm{tr}\left(t^a_{Ad}\, t^b_{Ad}\right) \tilde{A}_\mu(p)\, \tilde{A}_\nu(-p)\cdot\left\{ \int \frac{d^dk}{(2\pi)^d}\, \frac{\delta_{\mu\nu}}{k^2} + \right.$$

$$\left. -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\, \frac{(p+2k)^\mu (p+2k)^\nu}{k^2\,(k+p)^2} \right\}$$

Per calcolare $\{\ldots\}$ usiamo gli integrali di Feynman

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+a^2)^A} = \frac{\Gamma(A-d/2)}{(a^2)^{A-d/2}\,(4\pi)^{d/2}\,\Gamma(A)}$$

$$\Rightarrow \lim_{\epsilon\to 0} \int \frac{d^dk}{(2\pi)^d} \underbrace{\frac{1}{k^2+\epsilon^2}}_{A=1 \quad a=\epsilon} = \lim_{\epsilon\to 0} \frac{\Gamma(1-d/2)}{\epsilon^{1-d/2}\,(4\pi)^{d/2}} = 0 \quad \text{in } d>2$$

$$\frac{1}{k^2(k+p)^2} = \int_0^1 d\xi \frac{1}{\left[k^2(1-\cancel{\xi}) + (\cancel{k^2}+2kp+p^2)\xi\right]^2} = \int_0^1 d\xi \frac{1}{(k^2+2\xi p\cdot k+\xi p^2)^2}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+2kp+b^2)^A} = \frac{\Gamma(A-d/2)}{(b^2-p^2)^{A-d/2}\,(4\pi)^{d/2}\,\Gamma(A)} \qquad \Gamma(n+1)=n!$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{p^\mu p^\nu}{(k^2+2\xi pk+\xi p^2)^2} = \frac{p^\mu p^\nu\,\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}\,(4\pi)^{d/2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{(k^2+2kp+b^2)^A} = -\frac{p^\mu\,\Gamma(A-d/2)}{(b^2-p^2)^{A-d/2}\,(4\pi)^{d/2}\,\Gamma(A)}$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{2k^\mu p^\nu + 2k^\nu p^\mu}{(k^2+2\xi pk+\xi p^2)^2} = -\frac{4\xi\, p^\mu p^\nu\,\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}\,(4\pi)^{d/2}}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{(k^2+2kP+b^2)^A} = \frac{1}{(4\pi)^{d/2}\Gamma(A)}\left[\frac{p^\mu p^\nu\,\Gamma(A-\frac{d}{2})}{(b^2-p^2)^{A-d/2}} + \frac{1}{2}\delta^{\mu\nu}\frac{\Gamma(A-1-d/2)}{(b^2-p^2)^{A-1-d/2}}\right]$$

$$\Rightarrow \int \frac{d^dk}{(2\pi)^d} \frac{4k^\mu k^\nu}{(k^2+2\xi kp+\xi p^2)^2} = \frac{4\xi^2\,p^\mu p^\nu\,\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}(4\pi)^{d/2}} + \frac{1}{2}\frac{4\,\delta^{\mu\nu}\,\Gamma(1-d/2)}{(\xi(1-\xi)p^2)^{1-d/2}(4\pi)^{d/2}}$$

$$\Rightarrow -\frac{1}{2}\int \frac{d^dk}{(2\pi)^d} \frac{(p+2k)^\mu (p+2k)^\nu}{k^2 (k+p)^2} = -\frac{1}{2}\int_0^1 d\xi \left\{ \frac{(1-2\xi)^2 p^\mu p^\nu \Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-d/2}(4\pi)^{d/2}} + \frac{2\delta^{\mu\nu}\Gamma(1-d/2)}{(\xi(1-\xi)p^2)^{1-\frac{d}{2}}(4\pi)^{d/2}} \right\}$$

$4\xi^2 - 4\xi + 1 = (2\xi-1)^2$

espandiamo fns. in d attorno a $d=4$

$\Gamma(1-\frac{d}{2}) = \frac{1}{1-\frac{d}{2}}\Gamma(2-\frac{d}{2})$

$d = 2\omega \quad (\omega \to 2)$

Parte divergente in $d \to 4$:

$$\int_0^1 d\xi \, \frac{(1-2\xi)^2}{(4\pi)^2} p^\mu p^\nu \, \frac{1}{2-\omega}$$

$$\int_0^1 d\xi \,(1-4\xi+4\xi^2) = 1 - 2 + \frac{4}{3} = \frac{1}{3}$$

Parte divergente in $d \to 4$:

$$\int_0^1 d\xi \, \xi(1-\xi) \, \frac{\delta^{\mu\nu} p^2}{(4\pi)^2}\left(-\frac{2}{2-\omega}\right)$$

$$\int_0^1 \xi(1-\xi)\, d\xi = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$\leadsto \text{Parte div.} = -\frac{1}{2}\frac{1}{48\pi^2}\frac{1}{2-\omega}(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$\Downarrow$

$$\mathrm{Tr}\log\Delta_{gh} \underset{\text{parte divergente}}{\approx} \int \frac{d^dp}{(2\pi)^p} \, \mathrm{tr}(t^a_{Ad} t^b_{Ad}) \tilde{A}^{\mu a}(p)\tilde{A}^{\nu b}(-p)\left(-\frac{1}{2}\frac{1}{3}\frac{1}{16\pi^2}\frac{1}{2-\omega}(p^\mu p^\nu - p^2\delta^{\mu\nu})\right) + \dots$$

$$= -\frac{1}{2}\frac{1}{3}\frac{1}{(4\pi)^2} \underbrace{\mathrm{tr}(t^a_{Ad} t^b_{Ad})}_{C(Adj)\delta^{ab} \,=\, C_2(G)} \underbrace{\int \frac{d^dp}{(2\pi)^d} \tilde{A}^{\mu a}(p)\tilde{A}^{\nu b}(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})}_{} \frac{1}{2-\omega}$$

Ora vedremo che qta è l'espressione di $g^2 S[A]$ (la parte quadratica in A) nello spazio dei momenti

Parte quadratica di azione S:

$$\frac{1}{4g^2}\int (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})\, d^dx =$$

$$= \frac{1}{4g^2}\int d^dx \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\left(-iq_{1\mu}\tilde{A}_\nu^a(q_1) + iq_{1\nu}\tilde{A}_\mu^a(q_1)\right)\left(-iq_2^\mu \tilde{A}^{a\nu}(q_2) + iq_2^\nu \tilde{A}^{a\mu}(q_2)\right) \cdot e^{-iq_1x - iq_2x}$$

$\rightarrow \delta(q_1+q_2) \qquad q_1 = -q_2 \equiv p$

$$= \frac{1}{4g^2}\int \frac{d^dp}{(2\pi)^d}\left(p_\mu \tilde{A}_\nu^a(p) - p_\nu \tilde{A}_\mu^a(p)\right)\left(p^\mu \tilde{A}^{a\nu}(-p) - p^\nu \tilde{A}^{a\mu}(-p)\right) =$$

$$= -\frac{1}{2g^2}\int \frac{d^dp}{(2\pi)^d}\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)\tilde{A}_\mu^a(p)\,\tilde{A}_\nu^a(-p)$$

Contributo di Δ_{gauge}

$$\Delta^{\mu\nu}_{gauge} = -D^2\delta^{\mu\nu} - 2i F^{a\mu\nu} t^a_{Ad} = \underbrace{\Delta_{gh}}\,\delta^{\mu\nu} - 2iF^{a\mu\nu}t^a_{Ad}$$

quando tracciamo sugli indici di Lorentz, otteniamo un fattore 4. ← op. come nei ghost ma ora abbiamo $\delta^{\mu\nu}$

$$\mathrm{Tr}\log\Delta_{gauge} = \mathrm{Tr}\log\Big(-\partial^2\delta^{\mu\nu} + \overbrace{(\Delta_1+\Delta_2)\delta^{\mu\nu} - 2iF^{\mu\nu a}t^a_{Ad}}^{(\dots)}\Big) =$$

$$= \mathrm{Tr}\log(-\partial^2\delta^{\mu\nu}) + \mathrm{Tr}\log\left(1 + (-\partial^2)^{-1}(\dots)\right) \simeq$$

cost. trascurabile

termini con Δ_1, Δ_2 ricostruiscono $\mathrm{Tr}\log\Delta_{gh}\delta^{\mu\nu}$

$$\simeq \mathrm{Tr}\left[(-\partial^2)^{-1}(\dots)\right] - \frac{1}{2}\mathrm{Tr}\left[\left((-\partial^2)^{-1}(\dots)\right)^2\right] =$$

$F_{\mu\nu}t^a_{Ad}$ non contrib.

termini misti $F_{\mu\nu}\Delta_1\delta^{\mu\nu}$ sono zero perché $F_{\mu\nu}\delta^{\mu\nu} = 0$

$$= 4\,\mathrm{Tr}\log\Delta_{gh} + F^{\mu\nu}\text{-terms}$$

← quadratici in F

$$\text{div } F^{\mu\nu}\text{-terms} = \overbrace{-\frac{1}{2}(-2i)^2}^{+2} \mathrm{Tr}\left((-\partial^2)^{-1} F^{a,\mu\nu} t^a_{Ad} (-\partial^2)^{-1} F^b_{\nu\mu} t^b_{Ad}\right)$$

$$= -2 \int d^d y \langle y| (-\partial^2)^{-1} F^{\mu\nu a} \underset{\int d^d x |x\rangle\langle x|}{\wedge} (-\partial^2)^{-1} F^b_{\mu\nu} (-\partial^2)^{-1} |y\rangle \, \mathrm{tr}(t^a_{Ad} t^b_{Ad})$$

$$= -2 \int d^d y \, d^d x \, \underline{\langle y| (-\partial^2)^{-1} |x\rangle} \underline{\langle x | (-\partial^2)^{-1} |y\rangle} F^{\mu\nu a}(x) F^b_{\mu\nu}(y) \, \mathrm{tr}(t^a_{Ad} t^b_{Ad})$$

$$= -2 \int d^d y \, d^d x \, \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k'^2} F^{\mu\nu a}(x) F^b_{\mu\nu}(y) \, \mathrm{tr}(t^a t^b) e^{ik(y-x)} e^{ik'(x-y)}$$

transf. Fourier

$$= -2 \int d^d y \, d^d x \, \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{k^2} \frac{1}{k'^2} \mathrm{tr}(t^a t^b) e^{-ipx} e^{-iqy} e^{i(k-k')y} e^{i(k'-k)x} \cdot$$

$$\cdot (-i)^2 \left(p^\rho \tilde{A}^{a\sigma}(p) - p^\sigma \tilde{A}^{a\rho}(p) \right)\left(q_\rho \tilde{A}^b_\sigma(q) - q_\sigma \tilde{A}^b_\rho(q) \right)$$

$$\delta(k'-k-p) \qquad \delta(k-k'-q) \quad \rightarrow \quad \begin{array}{l} q = -p \\ k' = k+p \end{array}$$

$$= -\frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \tilde{A}^a_\mu(p) \tilde{A}^b_\nu(-p) \, \mathrm{tr}(t^a t^b) \cdot \int \frac{d^d k}{(2\pi)^d} \frac{4(p^\rho \delta^{\mu\sigma} - p^\sigma \delta^{\mu\rho})(p_\rho \delta^\nu_\sigma - p_\sigma \delta^\nu_\rho)}{k^2 (k+p)^2}$$

$$= 4 \int \frac{d^d p}{(2\pi)^d} \tilde{A}^a_\mu(p) \tilde{A}^b_\nu(-p) \, \mathrm{tr}(t^a t^b) (p^\mu p^\nu - p^2 \delta^{\mu\nu}) \int_0^1 d\xi \frac{\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-\frac{d}{2}} (4\pi)^{d/2}}$$

$$\underset{\text{div.}}{\sim} \frac{4 \, C(\mathrm{Adj}) \delta^{ab}}{(4\pi)^2} \frac{1}{2-\omega} \int \frac{d^d p}{(2\pi)^d} \hat{A}^a_\mu(p) \hat{A}^b_\nu(-p) (p^\mu p^\nu - p^2 \delta^{\mu\nu})$$

$$\Downarrow$$

$$\mathrm{Tr} \log \Delta_{gauge} \underset{\substack{\text{divergent} \\ \text{part}}}{\approx} 4 \, \mathrm{Tr} \log \Delta_{gh} \Big|_{\substack{\text{divergent} \\ \text{part}}} +$$

$$+ \frac{4}{(4\pi)^2} C_2(G) \delta^{ab} \frac{1}{2-\omega} \int \frac{d^d p}{(2\pi)^d} \hat{A}^a_\mu(p) \hat{A}^b_\nu(-p) (p^\mu p^\nu - p^2 \delta^{\mu\nu})$$

$$= -\cancel{4}^{\,2}\,\frac{1}{3\cancel{6}}\,\frac{1}{(4\pi)^2}\,C_2(G)\,\delta^{ab}\,\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\,\hat{A}^a_\mu(p)\,\hat{A}^b_\nu(-p)\,\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$

$$+\frac{4}{(4\pi)^2}\,C_2(G)\,\delta^{ab}\,\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\,\hat{A}^a_\mu(p)\,\hat{A}^b_\nu(-p)\,\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$

$$= \frac{10}{3}\,\frac{1}{(4\pi)^2}\,C_2(G)\,\delta^{ab}\,\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\,\hat{A}^a_\mu(p)\,\hat{A}^b_\nu(-p)\,\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$