Esercizi su stima di Bayes

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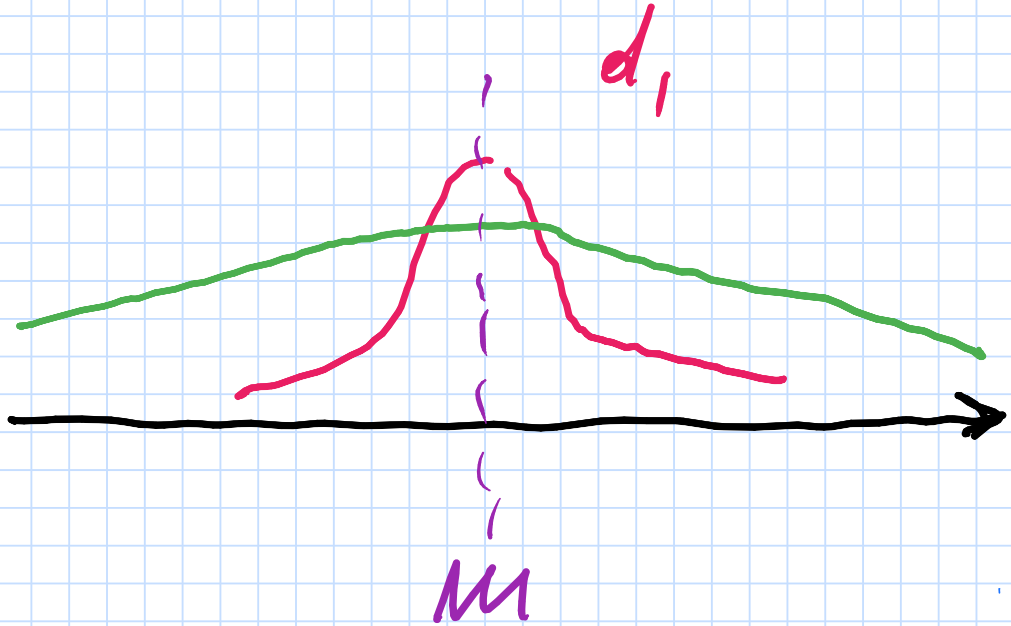
Es

2 v. a. indipendenti, gaussiane d_1, d_2

$d_1 \sim \mathcal{G}(m, 2)$

$d_2 \sim \mathcal{G}(m, 4)$



Stimare m con una osservazione di d_1 ed d_2

$$\hat{m} = \alpha d_1 + \beta d_2$$

① NO BIAS $\quad E(\hat{\mu}) = \mu$

$$E[\hat{\mu}] = \alpha E[d_1] + \beta E[d_2]$$

$$= (\alpha + \beta)\mu$$

NO BIAS $\Longleftrightarrow \alpha + \beta = 1$

② varianza minima

$$\hat{m} = \alpha d_1 + \beta d_2$$

$$\alpha + \beta = 1$$

$$\operatorname{var} \hat{m} = E\left[(\hat{m} - m)^2\right]$$

$$= E\left[\left(\alpha d_1 + \beta d_2 - \underbrace{1}_{\alpha+\beta} \cdot m\right)^2\right]$$

$$= E\left[\left(\alpha(d_1 - m) + \beta(d_2 - m)\right)^2\right] = \rightarrow$$

$$\rightarrow \alpha^2 \, \underline{E}\left[(d_1 - \mu)^2\right] + \beta^2 \, \underline{E}\left[(d_2 - \mu)^2\right]$$

$\hookrightarrow$ tutti gli altri termini valgono 0

$$= \alpha^2 \cdot 2 + \beta^2 4 = 2\alpha^2 + 4\beta^2$$

$$\alpha + \beta = 1 \rightarrow \beta = 1 - \alpha$$

$$\operatorname{var} \hat{\mu} = 6\alpha^2 - 8\alpha + 4$$

$$\mathrm{var}\,\hat{u} = 6\alpha^2 - 8\alpha + 4$$

$$\frac{d}{d\alpha}\left(\mathrm{var}\,\hat{u}\right) = 0 \longrightarrow 12\alpha - 8 = 0$$

$$\begin{cases} \alpha = 2/3 \\ \beta = 1/3 \end{cases}$$

$$\hat{u} = \frac{2}{3} d_1 + \frac{1}{3} d_2 \qquad \mathrm{var}\,\hat{u} = 4/3$$

Es ② consideriamo 2 v.a. x, w

$E(x) = 0$ $\quad E(w) = 0$ $\quad$ scorrelate

$\sigma_x^2 = 1$ $\quad \sigma_w^2 = 1$ $\quad E(xw) = 0$

Vorrei conoscere il valore attuale di x all'istante t (faccio un esperimento) $\rightarrow$ x non accessibile. Posso osservare un'altra v.a.

$$y = x + w$$

$\hat{x} = ?$ stimatore lineare ottimo

$$\hat{\gamma} = \frac{\Lambda_{\gamma d}}{\Lambda_{dd}}\, d$$

$\sigma_{\gamma d}$

σ_d^2

osservazione

$d \leftrightarrow y$

$\gamma \leftrightarrow x$

$\Lambda_{dd} \leftrightarrow \sigma_y^2$ ///

$\Lambda_{\gamma d} \leftrightarrow \sigma_{xy}$ ///

$$E(y) = ?$$

$$y = x + w$$

$$\longrightarrow E(y) = E(x) + E(w) = 0$$

$$\sigma_{xw} = 0$$

$$\sigma_y^2 = \sigma_x^2 + \sigma_w^2 = 2$$

$$\gamma_d = \sigma_{xy} = E(xy) \qquad y = x + w$$

$$E(xy) = E[x(x+w)] = \longrightarrow$$

$$E(xy) = E\left[x(x+w)\right] = \underbrace{E(x^2)}_{\sigma_x^2} + \underbrace{E(xw)}_{0}$$

$$= 1 = \sigma_{xy}$$

$$\hat{x} = \frac{\sigma_{xy}}{\sigma_{yy}} \cdot y = \frac{1}{2} y$$

$$\mathrm{var}(\theta - \hat{\theta}) = \sigma_{\theta\theta} - \frac{\sigma_{\theta d}^2}{\sigma_{dd}}$$

Supponiamo che le v.a. siano x, w, z IND.
e di poter fare 2 osservazioni differenti

$x \to E(x) = 0$
$\sigma^2_x = 1$

$w \to E(w) = 0$
$\sigma^2_w = 1$

$z \to E(z) = 0$
$\sigma^2_z = \lambda^2$

$$\begin{cases} y_1 = x + w \\ y_2 = x + z \end{cases}$$

$$\hat{\delta} = \Lambda_{\delta d} \cdot \Lambda_{dd}^{-1} \cdot d$$

$$\hat{x} = \Lambda_{xd} \cdot \Lambda_{dd}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Lambda_{dd} = \begin{bmatrix} \sigma_{y_1}^2 & \sigma_{y_1 y_2} \\ \sigma_{y_2 y_1} & \sigma_{y_2}^2 \end{bmatrix}$$

$$\sigma_{y_1}^2 = 2$$

$$\sigma_{y_2}^2 = 1 + d^2$$

$$\sigma_{y_1 y_2} = E\left[y_1 \; y_2\right] = E\left[(x+w)(x+z)\right]$$

$$= E(x^2) + \cancel{E(xw)} + \cancel{E(xz)} + \cancel{E(wz)}$$

$$0 \qquad 0 \qquad 0$$

$$= 1$$

$$\Lambda_{old} = \begin{bmatrix} 2 & 1 \\ 1 & 1+d^2 \end{bmatrix}$$

$$\Lambda_{dd}^{-1} = \begin{bmatrix} \frac{d^2+1}{2d^2+1} & -\frac{1}{2d^2+1} \\ -\frac{1}{2d^2+1} & \frac{2}{2d^2+1} \end{bmatrix}$$

$$\Lambda_{d\theta} = \begin{bmatrix} E(y_1 x) \\ E(y_2 x) \end{bmatrix}$$

$$\Lambda_{\theta d} = \Lambda_{d\theta}^T$$

$$E(y_1 x) = E[(x+w)x] = 1$$

$$E(y_2 x) = E[(x+z)x] = 1$$

$$\Lambda_{d\sigma} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Lambda_{\sigma d} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\hat{x} = \Lambda_{xd} \cdot \Lambda_{dd}^{-1} \cdot d = [1 \quad 1]\, \Lambda_{dd}^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2d^2+1} \left(y_2 + d^2 y_1 \right)$$

se $d^2 \gg 1 \implies \hat{x} \longrightarrow \frac{1}{2} y_1$

se $d^2 \ll 1 \implies \hat{x} \longrightarrow y_2$

¿ Se $y_2 = z$?

$$\begin{cases} y_1 = x + w \\ y_2 = z \end{cases}$$

$E(z) = 0$

$\sigma_z^2 = 1$

$E(xw) = 0$

$E(xz) = 0$

$E(wz) = 0$

$$\Lambda_{dd} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Lambda_{d\sigma} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{x} = \Lambda_{0d}\cdot\Lambda_{dd}^{-1}\cdot d$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{1}{2} y_1$$

$$\mathrm{var}(x - \hat{x}) = \sigma_x^2 - \Lambda_{\sigma d} \Lambda_{dd}^{-1} \Lambda_{d\sigma}$$

$$= 1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= 1/2$$