Nov 18

<u>Theorem</u> E B space, E', $\overline{D_{E'}(0,1)} = \{ f \in E' : \|f\|_{E'} \le 1 \}$.

For the $\sigma(E', E)$ top, $\overline{D_{E'}(0,1)}$ is compact

<u>Pf</u> $\Phi : E' \longrightarrow \mathbb{R}^E = \{ f : (f : E \to \mathbb{R}) \} = \{ \{f(x)\}_{x \in E} \}$ with product top

$f \in E' \longrightarrow (f(x))_{x \in E}$

We will show that

$\phi : E' \xrightarrow{\sigma(E', E)} \phi(E)$ is homeomorphism.

The fact that is bijective is clear.

$* \ \phi : (E', \sigma(E', E)) \longrightarrow \mathbb{R}^E$ is continuous

iff

$$\begin{array}{c} \mathbb{R}^E \\ \big\downarrow ev_x \\ \mathbb{R} \end{array}$$

$ev_x \circ \phi$ is continuous for any x

$ev_x \circ \phi(f) = f(x) \qquad x \in E$

$$\left(E', \sigma(E',E)\right) \longrightarrow \mathbb{R}$$

$$f \longrightarrow f(x)$$

is continuous. So $*$ is continuous.

$$\begin{tikzcd}
\phi^{-1} : \phi(E) \arrow[r] \arrow[dr, dashed, "ev_x \circ \phi^{-1}"'] & \left(E', \sigma(E',E)\right) \arrow[d, "ev_x"] \\
& \mathbb{R}
\end{tikzcd}$$

Need to show that

$ev_x \circ \phi^{-1} : \phi(E) \longrightarrow \mathbb{R}$ is continuous
$w \in$

$$ev_x \circ \phi^{-1}(w) = w(x) = ev_x\, w$$

By the continuity of $ev_x : \mathbb{R}^E \longrightarrow \mathbb{R}$

it follows that $ev_x \circ \phi^{-1}$ is continuous $\forall x \in E$.

$$(1) \quad \phi\left(\overline{D_{E'}(0,1)}\right) = \phi\left(\{f \in E' : \|f\|_{E'} \leq 1\}\right)$$

It is enough to show that this subspace of $\mathbb{R}^E$ is compact

We claim that the set in (1)

$K_1 \cap K_2$

$K_1 = \{ w \in \mathbb{R}^E : |w(x)| \leq \|x\|_E \ \forall x \in E \}$

$K_2 = \{ w \in \mathbb{R}^E ; \quad w(x+y) = w(x) + w(y) \quad \forall x, y \in E$
$\qquad w(\lambda x) = \lambda w(x) \qquad \forall \lambda \in \mathbb{R}, \forall x \in E \}$

K_2 is closed for fixed $x, y \in E$

$\{ w \in \mathbb{R}^E : \quad w(x+y) = w(x) + w(y) \}$ is closed

$w \to \quad w(x+y) - w(x) - w(y) = 0$

$\mathbb{R}^E \longrightarrow \mathbb{R}$

$\mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x_1, x_2, x_3) \to x_3 - x_1 - x_2$

K_1 is $\prod_{x \in E} [-\|x\|_E, \|x\|_E]$

is compact in $\mathbb{R}^E$

$\Rightarrow K_2 \cap K_1$ is compact.

$\phi(\overline{D_{E'}(0,1)})$ is compact in $\mathbb{R}^E$, in $\phi(E')$

Hence $\overline{D_{E'}(0,1)}$ is compact in

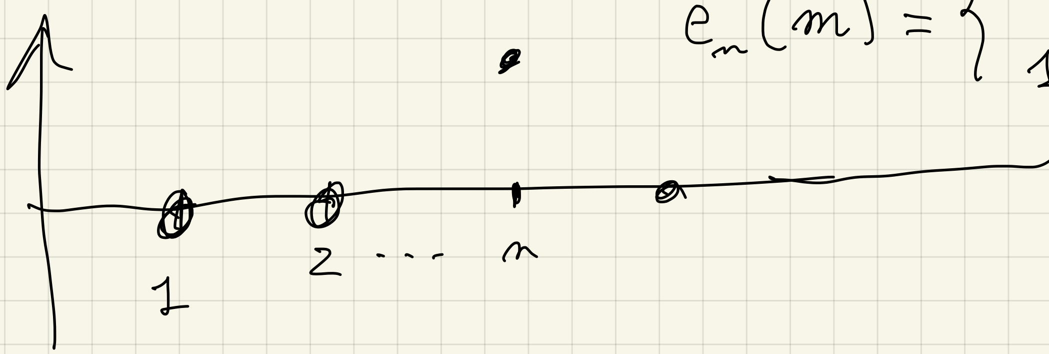
$(E', \sigma(E',E))$.

is an isometry

<u>Example</u> $E = \ell^\infty(\mathbb{N})$ $\qquad E' \supseteq \ell^1(\mathbb{N})$

E' $\qquad \{e_n\}$

$$e_n(m) = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$



$\|e_n\|_{E'} = \| e_n\|_{\ell^1} = 1$

Therefore $\{ e_n : n \in \mathbb{N} \}$ is relatively compact for the $\sigma(E',E)$ top.

Yet there are no convergent subsequences.

If $\{ e_{n_k} \}$ was a convergent subsequence for $\sigma(E',E)$, then there could be a limit for any $\xi \in E$ $\ell^\infty(\mathbb{N})$

of the sequence $\langle e_{n_k}, \xi \rangle_{E' \times E}$

$$\xi(m) = \begin{cases} 0 & m \neq n_k \quad \forall\ k \\ (-1)^k & m = n_k \end{cases}$$

$$\|\xi\|_{\ell^\infty} = 1$$

$$\langle e_{n_k}, \xi \rangle_{E' \times E} = 1 \cdot \xi(n_k) = (-1)^k$$

which has no limit in k.

Reflexive spaces

E B space is reflexive if $J : E \to E''$

is an isomorphism

Theorem (Kakutani) E is reflexive if and only if $\overline{D_E(0,1)}$ is compact for the $\sigma(E, E')$ top.

Pf If E is reflexive then $J\,\overline{D_E(0,1)} = \overline{D_{E''}(0,1)}$

where the latter is compact for $\sigma(E'', E')$ top.

We show that

$$
\begin{tikzcd}
J^{-1} : (E'', \sigma(E'',E')) \arrow[r] \arrow[dr] & (E, \sigma(E,E')) \arrow[d, "f"] \\
& \mathbb{R}
\end{tikzcd}
\qquad f \in E'
$$

is continuous

$\xi \in E''$

$$\xi \mapsto \langle J^{-1}\xi, f \rangle_{E \times E'} = \langle \xi, f \rangle_{E'' \times E'}$$

$\overline{D_E(0,1)}$ compact for $\sigma(E, E')$

$\Longrightarrow$ E is reflexive

<u>Lemma</u> (Goldstein) E B space. $J : E \to E''$

Then $J\, D_E(0,1)$ is dense in $D_{E''}(0,1)$

for $\sigma(E'', E')$ top.

$\overline{D_E(0,1)}$ compact for $\sigma(E, E')$ top

$J : E \to E''$ continuous strongly

continuous

$$J : (E, \sigma(E, E')) \longrightarrow (E'', \sigma(E'', E'''))$$

$$\Rightarrow J : (E, \sigma(E, E')) \longrightarrow (E'', \sigma(E'', E'))$$

continuously

$J \overline{D_E(0,1)}$ is compact in $(E'', \sigma(E'', E'))$

$J D_E(0,1)$ is dense in $D_{E''}(0,1)$ for the

$\sigma(E'', E')$

$$J \overline{D_E(0,1)} \supseteq D_{E''}(0,1)$$

$\Rightarrow J E = E'' \Rightarrow E$ is reflexive

Lemma E a B space

1) $M \subseteq E$ closed vector subspace

E reflexive $\Rightarrow$ M reflexive

2) E reflexive $\Leftrightarrow$ E' is reflexive

Pf 1) ~~At~~ In M the $\sigma(M, M')$ coincides with the topology induced by $(E, \sigma(E, E'))$.

$\overline{D_E(0,1)}$ is compact for $\sigma(E, E')$

$\Rightarrow \overline{D_M(0,1)}$ is a closed subset of $\overline{D_E(0,1)}$

$\Rightarrow \overline{D_M(0,1)}$ is compact for the top induced by $(E, \sigma(E, E'))$

2) Assume E reflexive, $E = E''$

By B-A $\overline{D_{E'}(0,1)}$ is compact for $\sigma(E', E) = \sigma(E', E'')$

$\Rightarrow E'$ is reflexive

Suppose instead that E' is reflexive

$\Rightarrow E''$ is reflexive

$JE \subseteq E''$ $\quad$ JE is closed in E''

$\Rightarrow$ JE is reflexive. $\Rightarrow$ E is reflexive

<u>Lemma</u> E B and reflexive and

$K \subset E$ bounded, strongly closed convex set

Then K is compact for $\sigma(E, E')$

<u>Pf</u> $K \subset \lambda \, \overline{D_E(0,1)} = \overline{D_E(0,\lambda)}$

compact in $\sigma(E, E')$

K is closed in $\sigma(E, E') \Rightarrow K$ compact for $\sigma(E, E')$

<u>Corollary</u> E B reflexive, $A \subseteq E$ closed convex. Let $\phi : A \to \mathbb{R}$ convex and lower semicontinuous and with

$$\cancel{\ } \lim_{\substack{\|x\|_E \to \infty}} \phi(x) = +\infty$$

Then $\exists$ a minimum point x_m for ϕ in A.

<u>Pf</u> For any value $\lambda_0 \in \phi(A)$

$$K_0 = \cancel{A \cap} \phi^{-1}((-\infty, \lambda_0]) = \phi^{-1}((-\infty, \lambda_0])$$

K_0 is bounded $\Rightarrow$ K_0 is compact

for $\sigma(E, E')$

$\{\lambda_n\}$ in $\phi(A)$

$$\lim_{n \to +\infty} \lambda_n = \inf \phi(A)$$

$$\lambda_n \downarrow \inf \phi(A) \qquad \phi(x_n) = \lambda_n$$

$$\cancel{x_n \in} K_n = \phi^{-1}((-\infty, \lambda_n])$$

$$K_n \supsetneq K_{n+1} \qquad K_n \neq \emptyset$$

$x_m \in \bigcap_{n \geq 1}^{\infty} K_n \Rightarrow$ is non empty

$$\phi(x_m) \leq \lambda_n \qquad \forall n$$

$$\Rightarrow \phi(x_m) = \inf \phi(A)$$