Corso di Laurea in Fisica - UNITS

Istituzioni di Fisica per il Sistema Terra

Approximate Solutions of the Navier-Stokes Equation

FABIO ROMANELLI

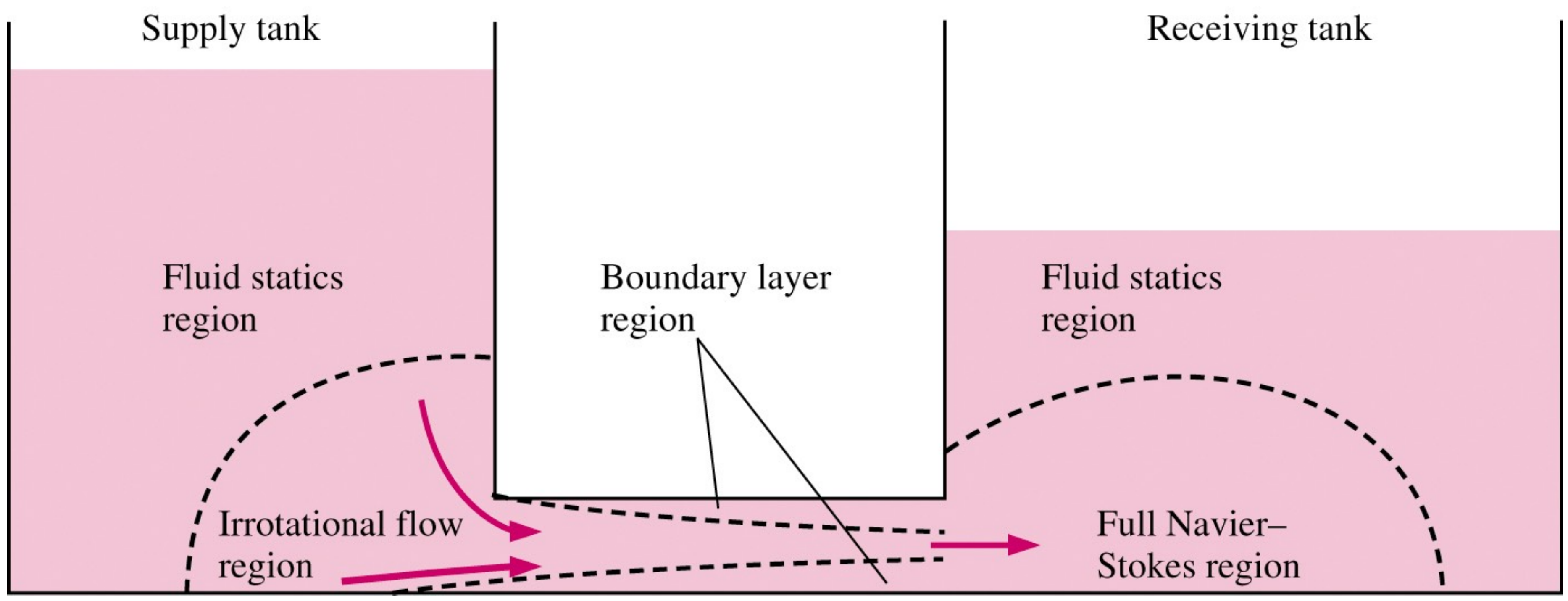
Department of Mathematics & Geosciences
University of Trieste
romanel@units.it

Objectives

- Appreciate why approximations are necessary, and know when and where to use.
- Understand effects of lack of inertial terms in the creeping flow approximation.
- Understand superposition as a method for solving potential flow.
- Predict boundary layer thickness and other boundary layer properties.

Introduction

- We derived the NSE and developed several exact solutions.
- We will study several methods for simplifying the NSE, which permit use of mathematical analysis and solution.
  - An approximate solution is one in which the Navier–Stokes equation is simplified in some region of the flow before we start the solution.
  - Term(s) are eliminated a priori depending on the class of problem, which may differ from one region of the flow to another.



Nondimensionalization of the NSE

- Purpose: Order-of-Magnitude analysis of the terms in the NSE, which is necessary for simplification and approximate solutions.
- We begin with the incompressible NSE

$$\rho \frac{D\vec{V}}{Dt} = \rho \left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V} \cdot \nabla \right) \vec{V} \right] = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

- Each term is dimensional, and each variable or property (ρ, V, t, μ, etc.) is also dimensional.
- What are the primary dimensions of each term in the NSE equation?

$$\text{Answer} : \left\{ \frac{m}{L^2 t^2} \right\}$$

Nondimensionalization of the NSE

- To nondimensionalize, we choose scaling parameters as follows

TABLE 10–1

Scaling parameters used to nondimensionalize the continuity and momentum equations, along with their primary dimensions

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	{L}
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

Nondimensionalization of the NSE

- Next, we define nondimensional variables, using the scaling parameters in Table 10-1

$$t^\star = ft \qquad \vec{x}^\star = \frac{\vec{x}}{L} \qquad \vec{V}^\star = \frac{\vec{V}}{V}$$

$$P^\star = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^\star = \frac{\vec{g}}{g} \qquad \nabla^\star = L\nabla$$

- To plug the nondimensional variables into the NSE, we need to first rearrange the equations in terms of the dimensional variables

$$t = \frac{1}{f}t^\star \qquad \vec{x} = L\vec{x}^\star \qquad \vec{V} = V\vec{V}^\star \qquad \nabla = \frac{1}{L}\nabla^\star$$

$$P = P_\infty + (P_0 - P_\infty)P^\star \qquad \vec{g} = g\vec{g}^\star$$

Nondimensionalization of the NSE

- Now we substitute into the NSE to obtain

$$\rho V f \frac{\partial \vec{V}^\star}{\partial t^\star} + \frac{\rho V^2}{L}\left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\frac{P_0 - P_\infty}{L}\nabla^\star P^\star + \rho g \vec{g}^\star + \frac{\mu V}{L^2}\nabla^{\star 2}\vec{V}^\star$$

- Every additive term has primary dimensions $\{m^1 L^{-2} t^{-2}\}$. To nondimensionalize, we multiply every term by $L/(\rho V^2)$, which has primary dimensions $\{m^{-1} L^2 t^2\}$, so that the dimensions cancel. After rearrangement:

$$\left[\frac{fL}{V}\right]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\left[\frac{P_0 - P_\infty}{\rho V^2}\right]\nabla^\star P^\star + \left[\frac{gL}{V^2}\right]\vec{g}^\star + \left[\frac{\mu}{\rho V L}\right]\nabla^{\star 2}\vec{V}^\star$$

Nondimensionalization of the NSE

- Each of terms in [ ] is a nondimensional group of parameters (Pi Group):

$$\left[\frac{fL}{V}\right]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -\left[\frac{P_0 - P_\infty}{\rho V^2}\right]\nabla^\star P^\star + \left[\frac{gL}{V^2}\right]\vec{g}^\star + \left[\frac{\mu}{\rho V L}\right]\nabla^{\star 2}\vec{V}^\star$$

Strouhal number

Euler number

Inverse of Froude number squared

Inverse of Reynolds number

$$[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star + \left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star$$

Navier-Stokes equation in nondimensional form

Nondimensionalization of the NSE

- Nondimensionalization vs. Normalization
  - NSE are now nondimensional, but not necessarily normalized. What is the difference?
  - Nondimensionalization concerns only the dimensions of the equation - we can use any value of scaling parameters L,V, etc.
  - Normalization is more restrictive than nondimensionalization. To normalize the equation, we must choose scaling parameters L,V, etc. that are appropriate for the flow being analyzed, such that all nondimensional variables are of order of magnitude unity, i.e., their minimum and maximum values are close to 1.0.

$$t^\star \sim 1 \qquad \vec{x}^\star \sim 1 \qquad \vec{V}^\star \sim 1 \qquad P^\star \sim 1 \qquad \vec{g}^\star \sim 1 \qquad \nabla^\star \sim 1$$

> If we have properly normalized the NSE, we can compare the relative importance of the terms in the equation by comparing the relative magnitudes of the nondimensional parameters St, Eu, Fr, and Re.

Comments about CD-NSE

- The nondimensionalized continuity equation contains no additional dimensionless parameters.
- The order of magnitude of the nondimensional variables is unity if they are nondimensionalized using a length, speed, frequency, etc., that are characteristic of the flow field.
  - The relative importance of the terms in depends only on the relative magnitudes of the dimensionless parameters
- Dynamic similarity between a model and a prototype requires all four of [ ] to be the same for the model and the prototype.
- If the flow is steady then the first term on the left side disappears. If the characteristic frequency f is very small such that St<<1, the flow is called quasi-steady.
- The effect of gravity is important only in flows with free-surface effects.
  - If no free surface the only effect of gravity on the flow dynamics is a hydrostatic pressure distribution in the vertical direction superposed on the pressure field due to the fluid flow.

Creeping Flow

- Also known as "Stokes Flow" or "Low Reynolds number flow"
- Occurs when Re << 1
  - ρ, V, or L are very small, e.g., micro-organisms, MEMS, nano-tech, particles, bubbles
  - μ is very large, e.g. honey, lava
    - g effect is negligible
    - Steady flow
    - Advective term is negligible

Creeping Flow

- To simplify NSE, assume St ~ 1, Fr ~ 1

$$[Eu]\,\nabla^{\star} P^{\star} = \left[\frac{1}{Re}\right] \nabla^{\star 2}\vec{V}^{\star}$$

Pressure forces

Viscous forces

- Since

$$P^{\star} \sim 1, \quad \nabla^{\star} \sim 1$$

$$Eu = \frac{P_0 - P_\infty}{\rho V^2} \sim \frac{1}{Re} = \frac{\mu}{\rho V L} \Longrightarrow P_0 - P_\infty \sim \frac{\mu V}{L}$$

Creeping Flow

- This is important $$P_0 - P_\infty \sim \frac{\mu V}{L}$$
  - Very different from inertia dominated flows where

$$P_0 - P_\infty \sim \rho V^2$$

  - Density has completely dropped out of NSE. To demonstrate this, convert back to dimensional form:

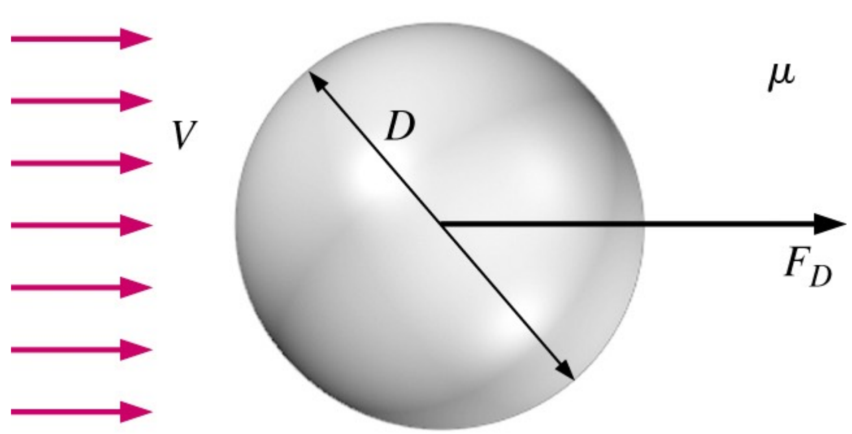
$$\nabla P = \mu \nabla^2 \vec{V}$$

  - This is now a LINEAR EQUATION which can be solved for simple geometries.

Creeping Flow

- Solution of Stokes flow is beyond the scope.

- Analytical solution for flow over a sphere gives a drag coefficient which is a linear function of velocity *V* and viscosity μ.



$$F_D = 3\pi\mu V D$$

Example

EXAMPLE 10–2 **Terminal Velocity of a Particle from a Volcano**

A volcano has erupted, spewing stones, steam, and ash several thousand feet into the atmosphere (Fig. 10–14). After some time, the particles begin to settle to the ground. Consider a nearly spherical ash particle of diameter 50 μm, falling in air whose temperature is -50°C and whose pressure is 55 kPa. The density of the particle is 1240 kg/m^3. Estimate the terminal velocity of this particle at this altitude.

Downward force:

$$F_{\text{down}} = W = \pi \frac{D^3}{6} \rho_{\text{particle}} g \qquad (1)$$

The aerodynamic drag force acting on the particle is obtained from Eq. 10–12, and the buoyancy force is the weight of the displaced air. Thus,

Upward force:

$$F_{\text{up}} = F_D + F_{\text{buoyancy}} = 3\pi\mu V D + \pi \frac{D^3}{6} \rho_{\text{air}} g \qquad (2)$$

We equate Eqs. 1 and 2, and solve for terminal velocity V,

$$V = \frac{D^2}{18\mu}(\rho_{\text{particle}} - \rho_{\text{air}})g$$

$$= \frac{(50 \times 10^{-6}\ \text{m})^2}{18(1.474 \times 10^{-5}\ \text{kg/m}\cdot\text{s})}[(1240 - 0.8588)\ \text{kg/m}^3](9.81\ \text{m/s}^2)$$

$$= \mathbf{0.115\ m/s}$$

Finally, we verify that the Reynolds number is small enough that creeping flow is an appropriate approximation,

$$\text{Re} = \frac{\rho_{\text{air}} V D}{\mu} = \frac{(0.8588\ \text{kg/m}^3)(0.115\ \text{m/s})(50 \times 10^{-6}\ \text{m})}{1.474 \times 10^{-5}\ \text{kg/m}\cdot\text{s}} = 0.335$$

- Although the equation for creeping flow drag on a sphere was derived for a case with Re<<1, it turns out that the approximation is reasonable up to Re ≅ 1.
- A more involved calculation, including a Reynolds number correction and a correction based on the mean free path of air molecules, yields a terminal velocity of 0.110 m/s; the error of the creeping flow approximation is less than 5 percent.

Inviscid Regions of Flow

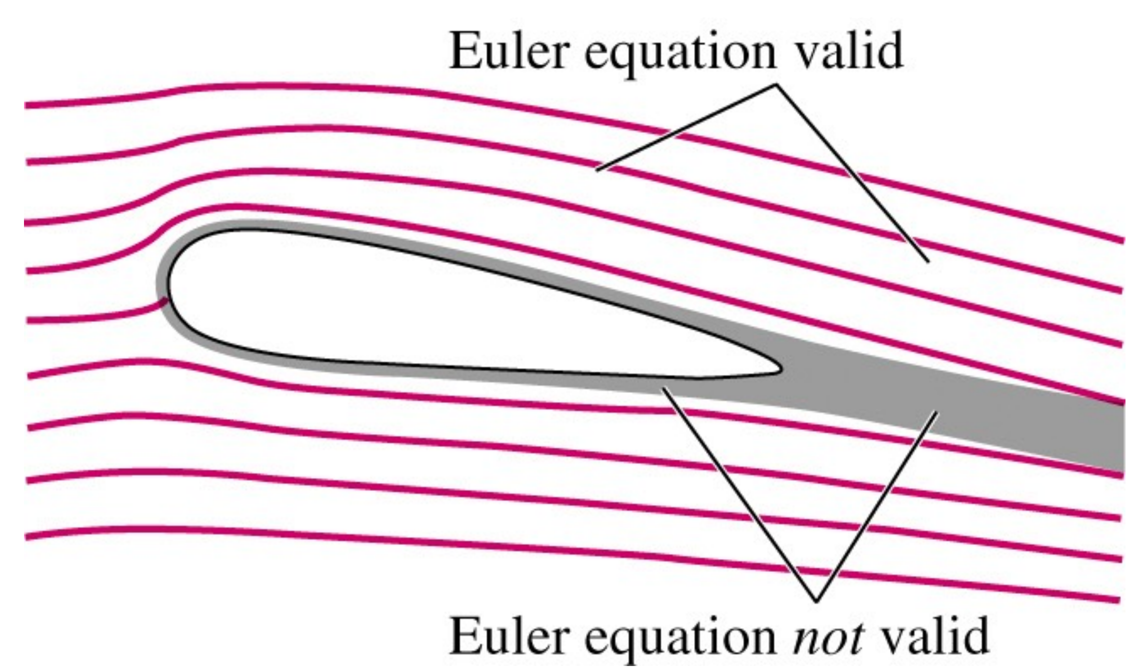
- Definition: Regions where net viscous forces are negligible compared to pressure and/or inertia forces

~0 if Re large

$$[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\,\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star + \left[\frac{1}{Re}\right]\nabla^{\star 2}\vec{V}^\star$$

Euler Equation

The Euler equation approximation is appropriate in high Reynolds number regions of the flow, where net viscous forces are negligible, far away from walls and wakes.



Inviscid Regions of Flow

- Euler equation often used in aerodynamics
- Elimination of viscous term changes PDE from mixed elliptic-hyperbolic to hyperbolic. This affects the type of analytical and computational tools used to solve the equations.
- Must "relax" wall boundary condition from no-slip to slip

<u>No-slip BC</u>

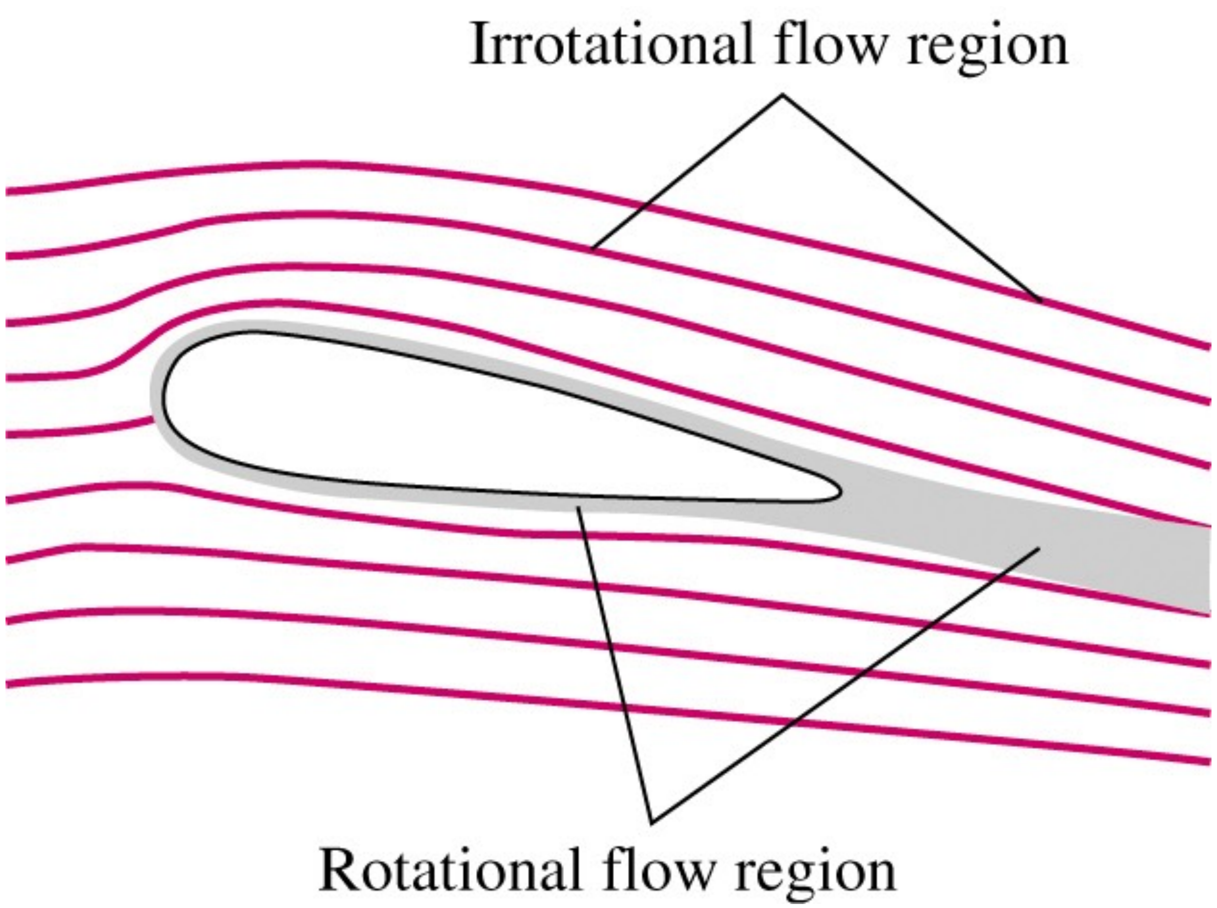
$u = v = w = 0$

<u>Slip BC</u>

$\tau_w = 0, V_n = 0$

V_n = normal velocity

Irrotational Flow Approximation



- Irrotational approximation: vorticity is negligibly small

$$\vec{\zeta} = \nabla \times \vec{V} \cong 0$$

- In general, inviscid regions are also irrotational, but there are situations where inviscid flow are rotational, e.g., solid body rotation

Irrotational Flow Approximation

- What are the implications of irrotational approximation. Look at continuity and momentum equations.
- Continuity equation
  - Use the vector identity $\nabla \times \nabla\phi = 0$
  - Since the flow is irrotational $\nabla \times \vec{V} = 0$

$$\boxed{\vec{V} = \nabla\phi}$$

ϕ is a scalar potential function

Irrotational Flow Approximation

- Therefore, regions of irrotational flow are also called regions of potential flow.
- From the definition of the gradient operator ∇

Cartesian

$$U = \frac{\partial \phi}{\partial x}, \quad V = \frac{\partial \phi}{\partial y}, \quad W = \frac{\partial \phi}{\partial z}$$

Cylindrical

$$U_r = \frac{\partial \phi}{\partial r}, \quad U_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}, \quad U_z = \frac{\partial \phi}{\partial z}$$

- Substituting into the continuity equation gives

$$\nabla \cdot \vec{V} = \nabla \cdot \nabla \phi = \boxed{\nabla^2 \phi = 0}$$

Irrotational Flow Approximation

- This means we only need to solve 1 linear scalar equation to determine all 3 components of velocity!

$$\nabla^2 \phi = 0$$

Laplace Equation

- Luckily, the Laplace equation appears in numerous fields of science, engineering, and mathematics. This means there are well developed tools for solving this equation.

Irrotational Flow Approximation

- Momentum equation
  - If we can compute ϕ from the Laplace equation (which came from continuity) and then velocity from the definition $\vec{V} = \nabla\phi$, why do we need the NSE? $\Rightarrow$ To compute Pressure.
  - To begin analysis, apply irrotational approximation to viscous term of the NSE:

$$\mu\nabla^2\vec{V} = \mu\nabla^2(\nabla\phi) = \mu\nabla(\underbrace{\nabla^2\phi}_{=0}) = 0$$

Irrotational Flow Approximation

- Therefore, the NSE reduces to the Euler equation for irrotational flow:

nondimensional

$$[St]\frac{\partial \vec{V}^\star}{\partial t^\star} + \left(\vec{V}^\star \cdot \nabla^\star\right)\vec{V}^\star = -[Eu]\,\nabla^\star P^\star + \left[\frac{1}{Fr^2}\right]\vec{g}^\star$$

dimensional

$$\rho\left[\frac{\partial \vec{V}}{\partial t} + \underbrace{\left(\vec{V}\cdot\nabla\right)\vec{V}}\right] = -\nabla P + \rho\vec{g}$$

- Instead of integrating to find P, use vector identity to derive Bernoulli equation

$$\left(\vec{V}\cdot\nabla\right)\vec{V} = \nabla\left(\frac{V^2}{2}\right) - \vec{V}\times\left(\nabla\times\vec{V}\right) = \nabla\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta}$$

Irrotational Flow Approximation

- This allows the steady Euler equation to be written as

$$= -g\vec{k} = \nabla(gz)$$

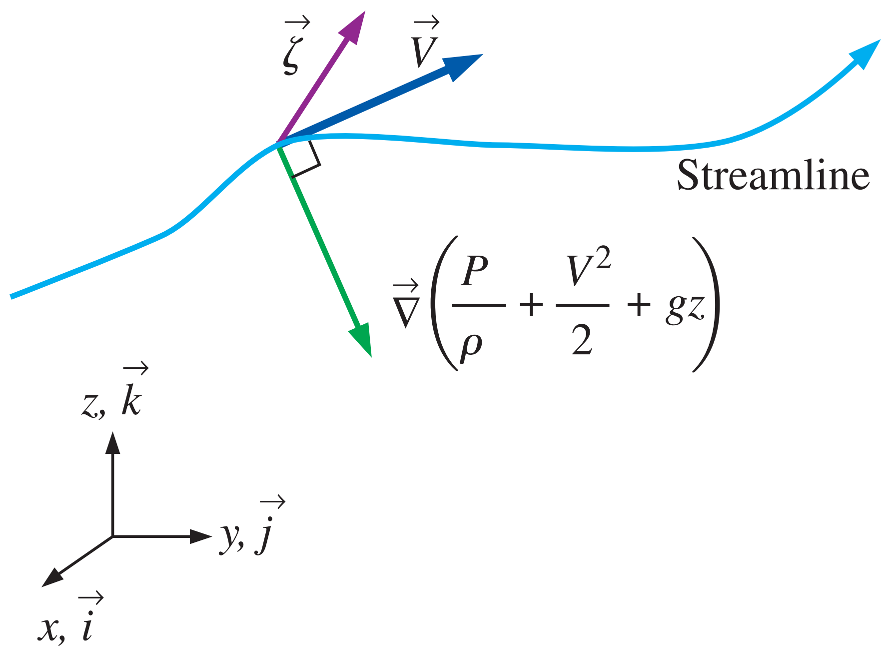
$$\nabla\left(\frac{V^2}{2}\right) - \vec{V}\times\vec{\zeta} = -\frac{1}{\rho}\nabla P + \vec{g}$$

$$\nabla\left(\frac{P}{\rho} + \frac{V^2}{2} + gz\right) = \vec{V}\times\vec{\zeta}$$

- This form of Bernoulli equation is valid for inviscid (and irrotational) flow since we've shown that NSE reduces to the Euler equation.

Irrotational Flow Approximation

- However



Inviscid

$$\frac{P}{\rho}+\frac{V^2}{2}+gz=C \quad \textbf{along a streamline}$$

Irrotational (ζ = 0)

$$\frac{P}{\rho}+\frac{V^2}{2}+gz=C \quad \textbf{everywhere}$$

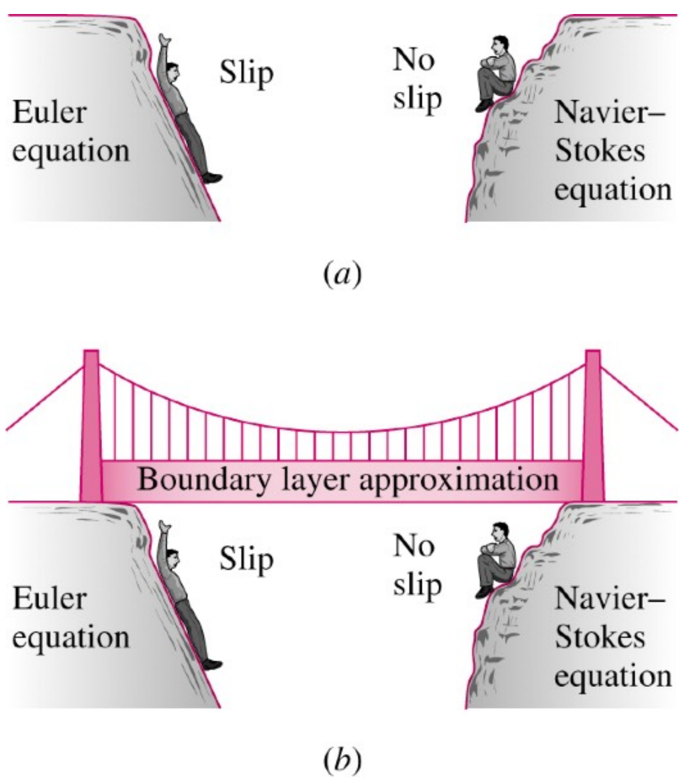
Irrotational Flow Approximation

- Therefore, the process for irrotational flow
  - Calculate ϕ from Laplace equation (from continuity)
  - Calculate velocity from definition $\vec{V} = \nabla\phi$
  - Calculate pressure from Bernoulli equation (derived from momentum equation)

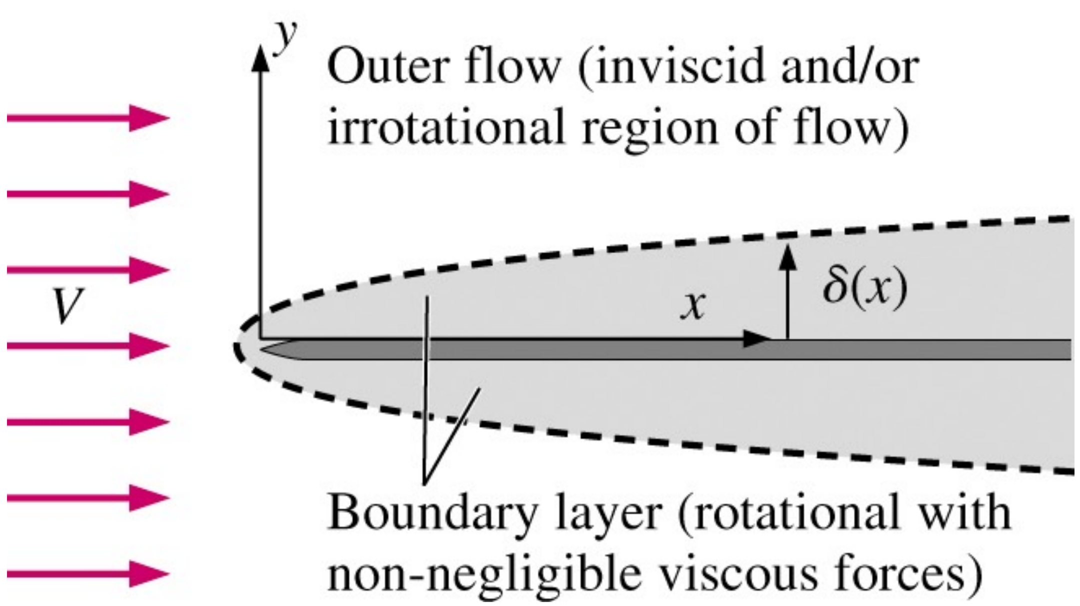
$$P = P_\infty + \rho\left[\frac{V_\infty^2 - V^2}{2} + g\left(z_0 - z\right)\right]$$

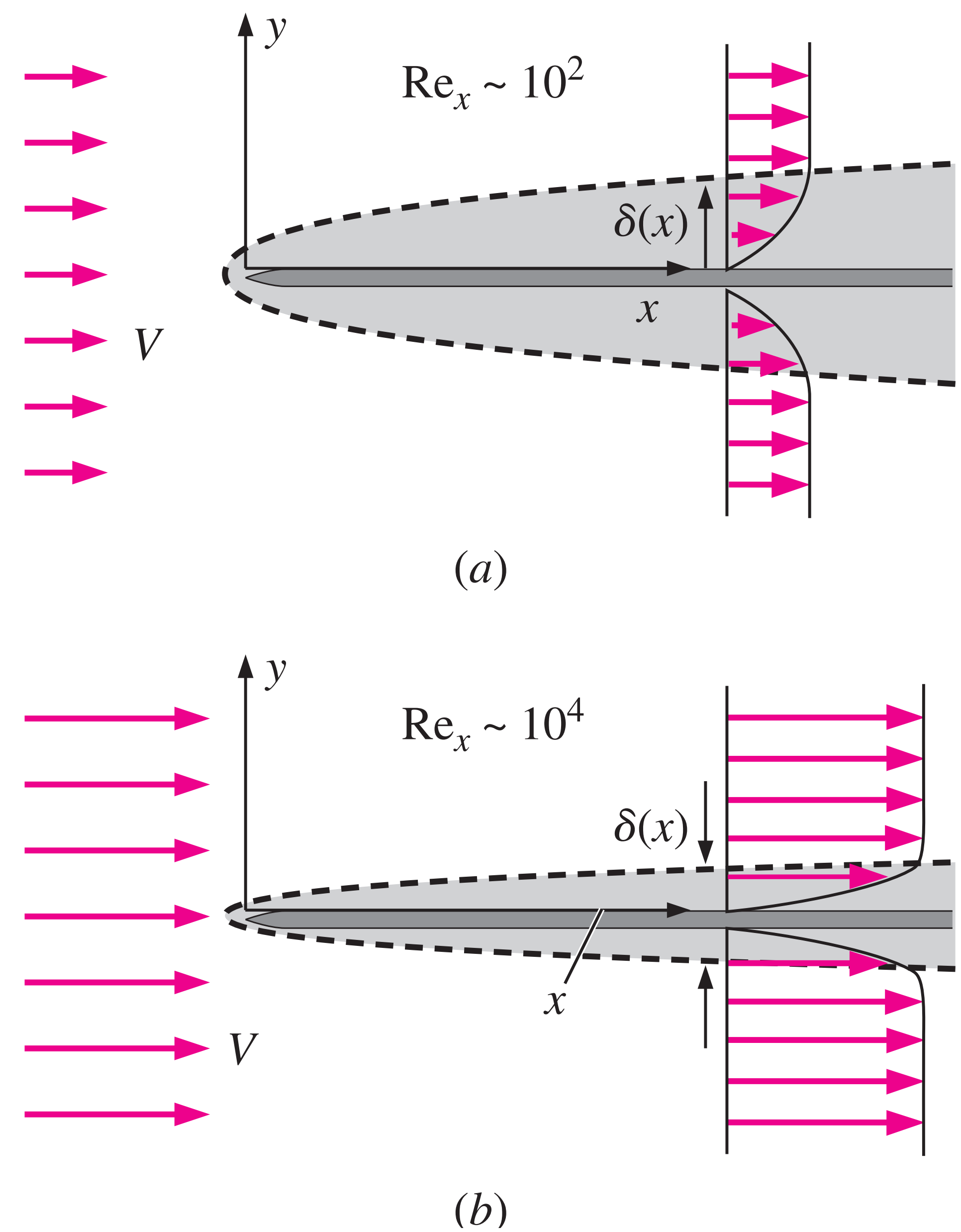
Valid for 3D or 2D

Boundary Layer (BL) Approximation



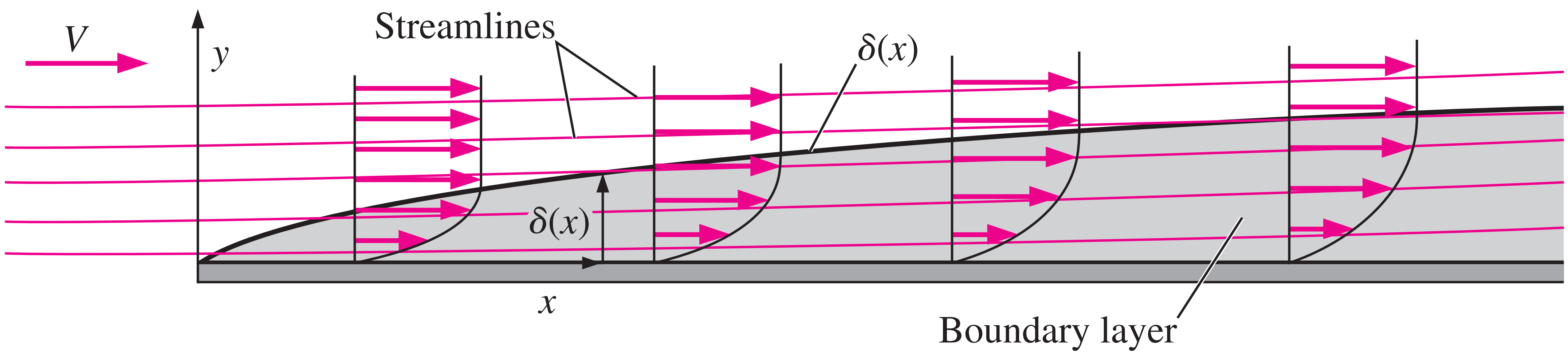
- BL approximation bridges the gap between the Euler and NS equations, and between the slip and no-slip BC at the wall.
- Prandtl (1904) introduced the BL approximation
- At a given x-location, the higher the Reynolds number, the thinner the boundary layer.





Boundary Layer (BL) Approximation

Flat Boundary Layer

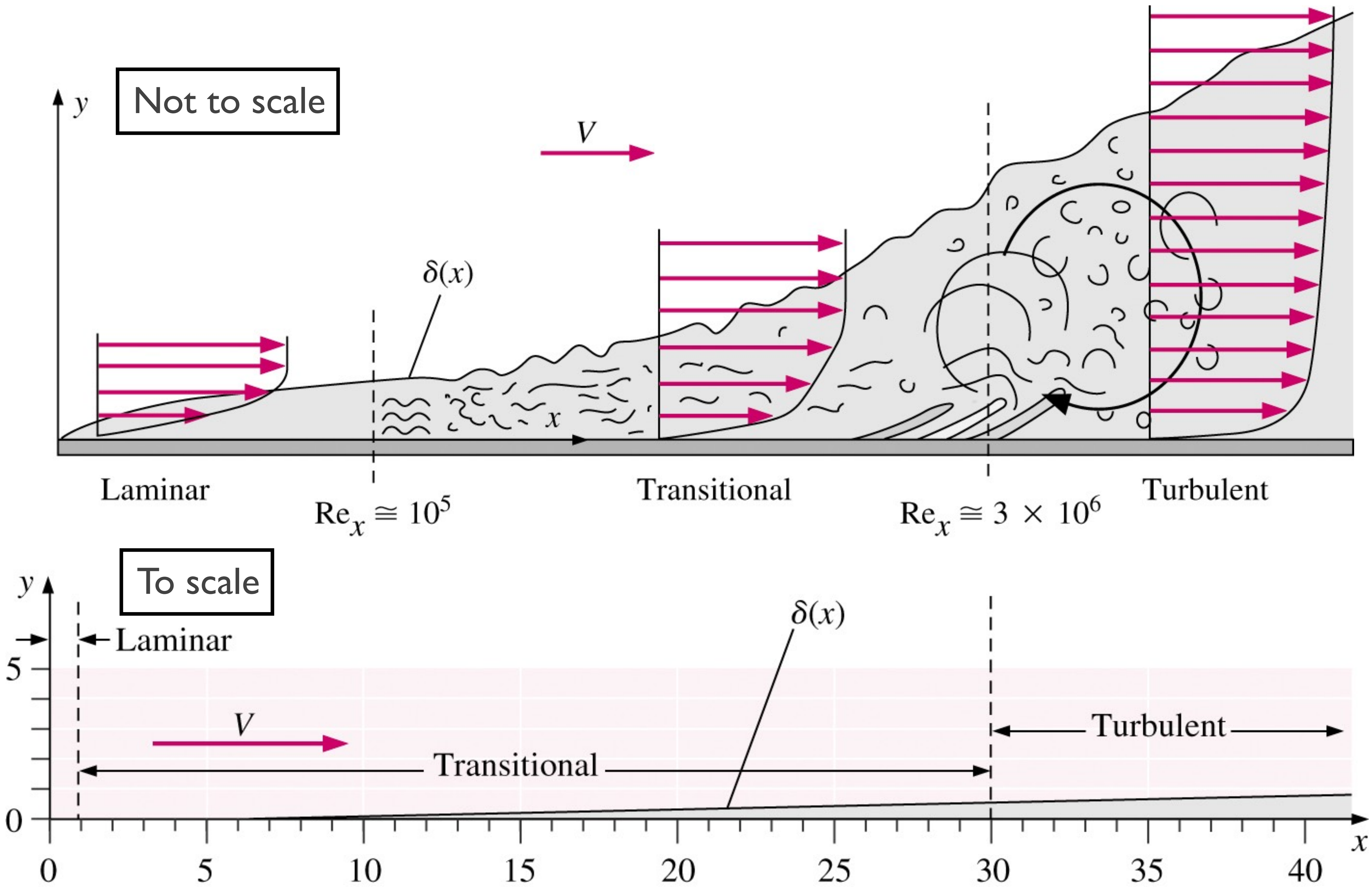


δ is proportional to the square root of Re_x. These results are valid only for a laminar boundary layer on a flat plate.

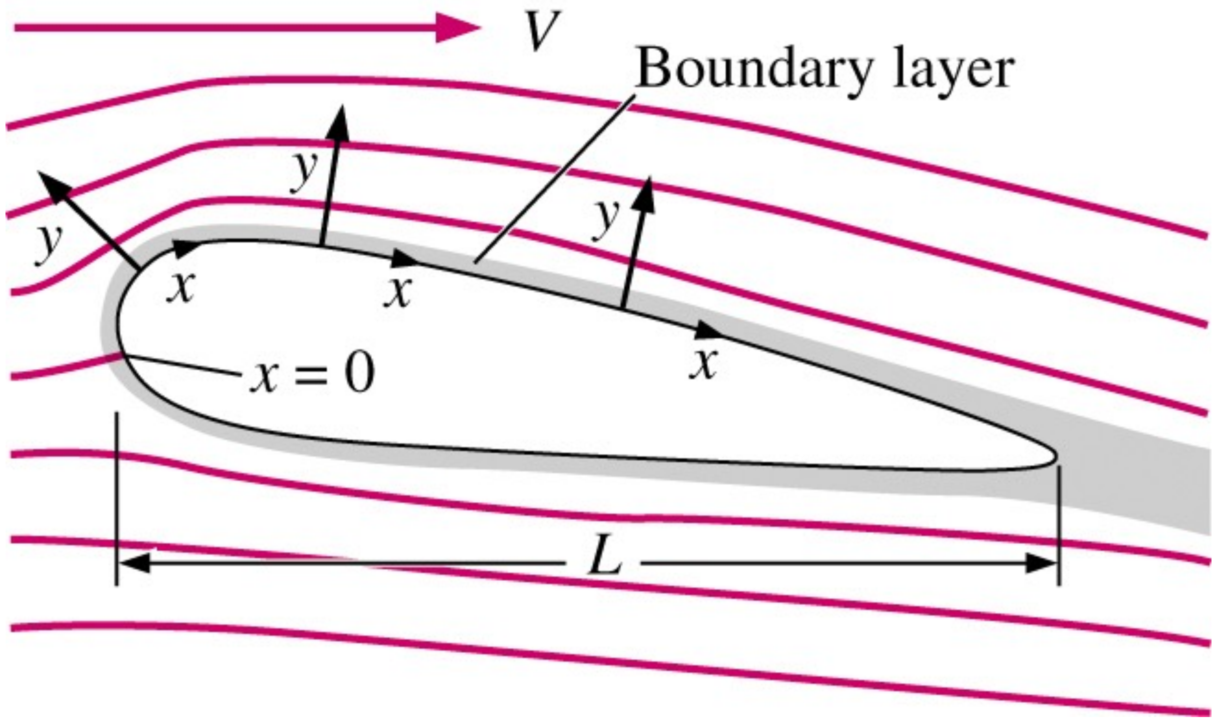
As we move down the plate to larger and larger values of x, Re_x increases linearly with x. At some point, infinitesimal disturbances in the flow begin to grow, and the boundary layer cannot remain laminar—it begins a transition process toward turbulent flow.

For a smooth flat plate with a uniform free stream, the transition process begins at a critical Reynolds number, $Re_{x,\,critical} \cong 10^5$, and continues until the boundary layer is fully turbulent at the transition Reynolds number, $Re_{x,\,transition} \cong 3 \cdot 10^6$

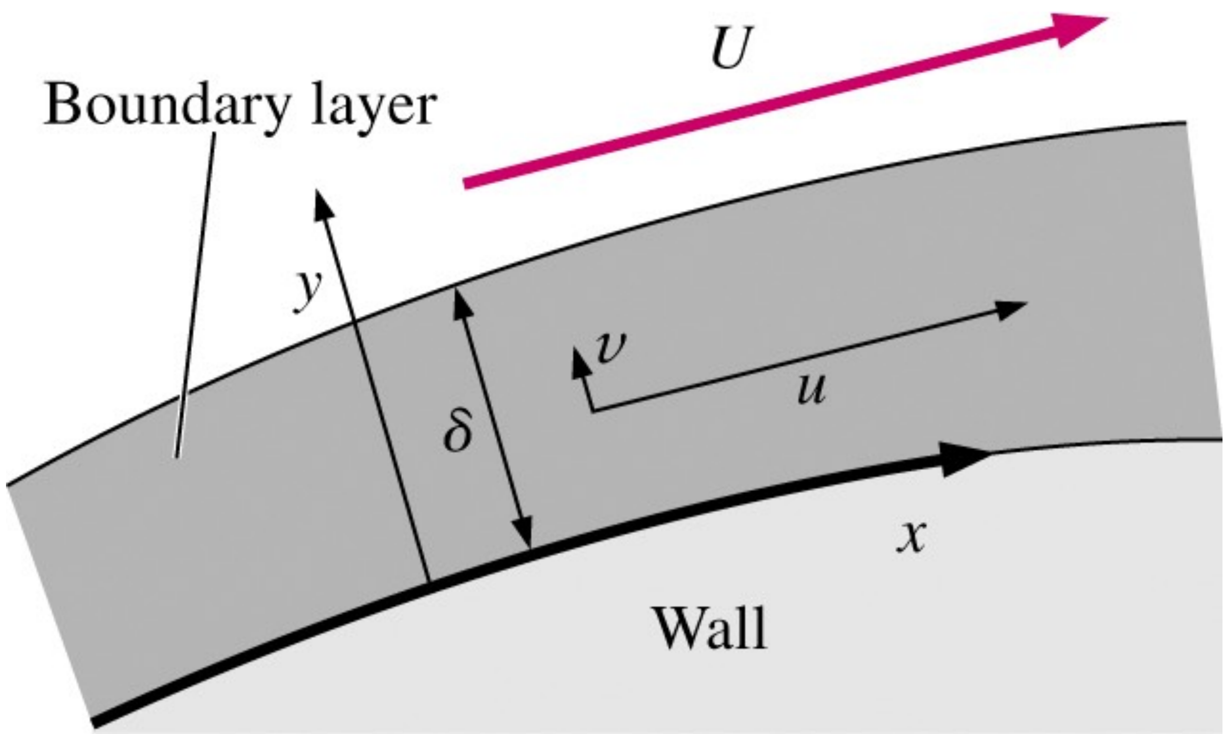
Boundary Layer (BL) Approximation



Boundary Layer (BL) Approximation



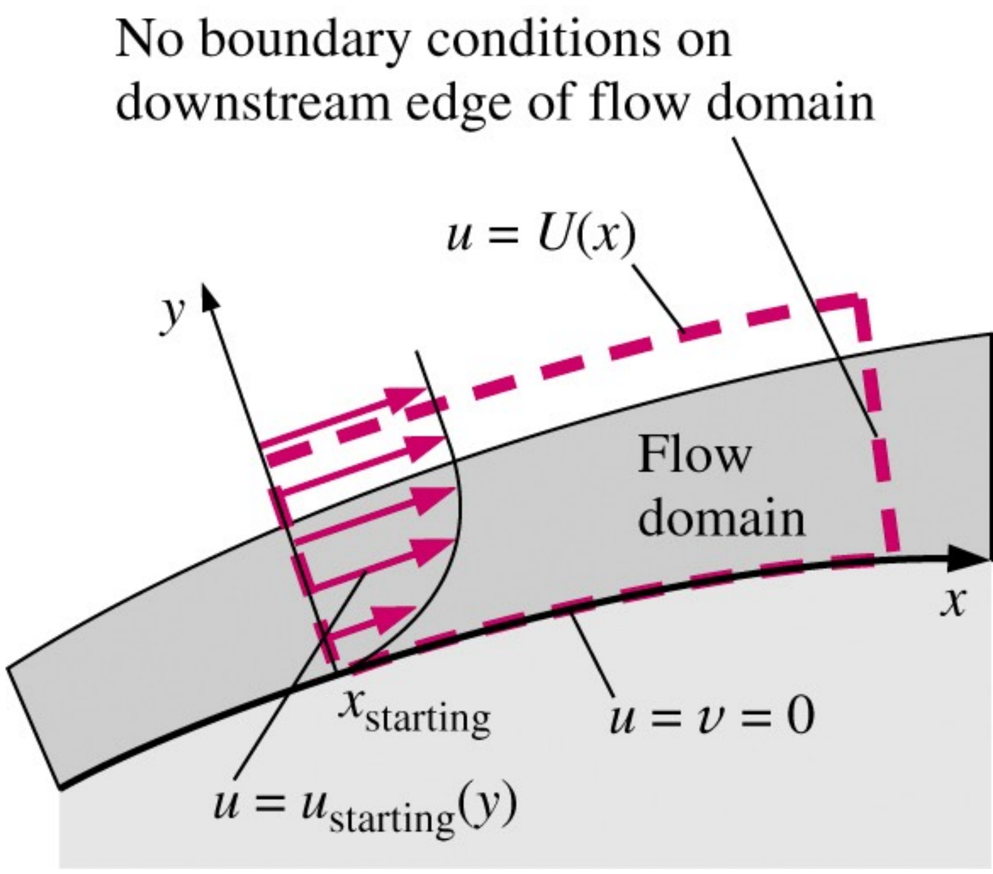
- BL Equations: we restrict attention to steady, 2D, laminar flow (although method is fully applicable to unsteady, 3D, turbulent flow)



- BL coordinate system
  - x : tangential direction
  - y : normal direction

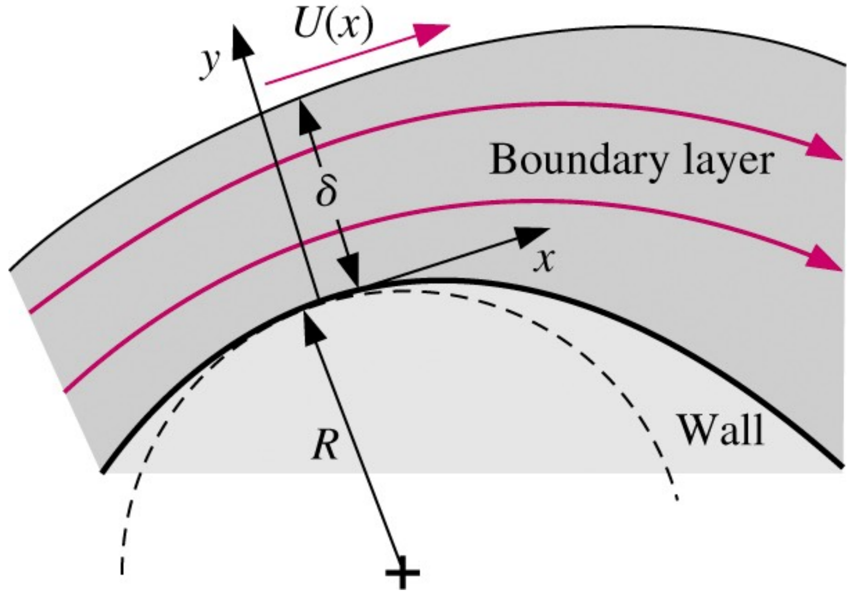
Boundary Layer Procedure

1. Solve for outer flow, ignoring the BL. Use potential flow (irrotational approximation) or Euler equation
2. Assume $\delta/L \ll 1$ (thin BL)
3. Solve BLE
   1. $y = 0 \Rightarrow$ no-slip, $u=0$, $v=0$
   2. $y = \delta \Rightarrow U = U_e(x)$
   3. $x = x_0 \Rightarrow u = u(x_0)$, $v=v(x_0)$
4. Calculate δ, θ, δ^*, τ_w, Drag
5. Verify $\delta/L \ll 1$
6. If δ/L is not $\ll 1$, use δ^* as body and goto step 1 and repeat

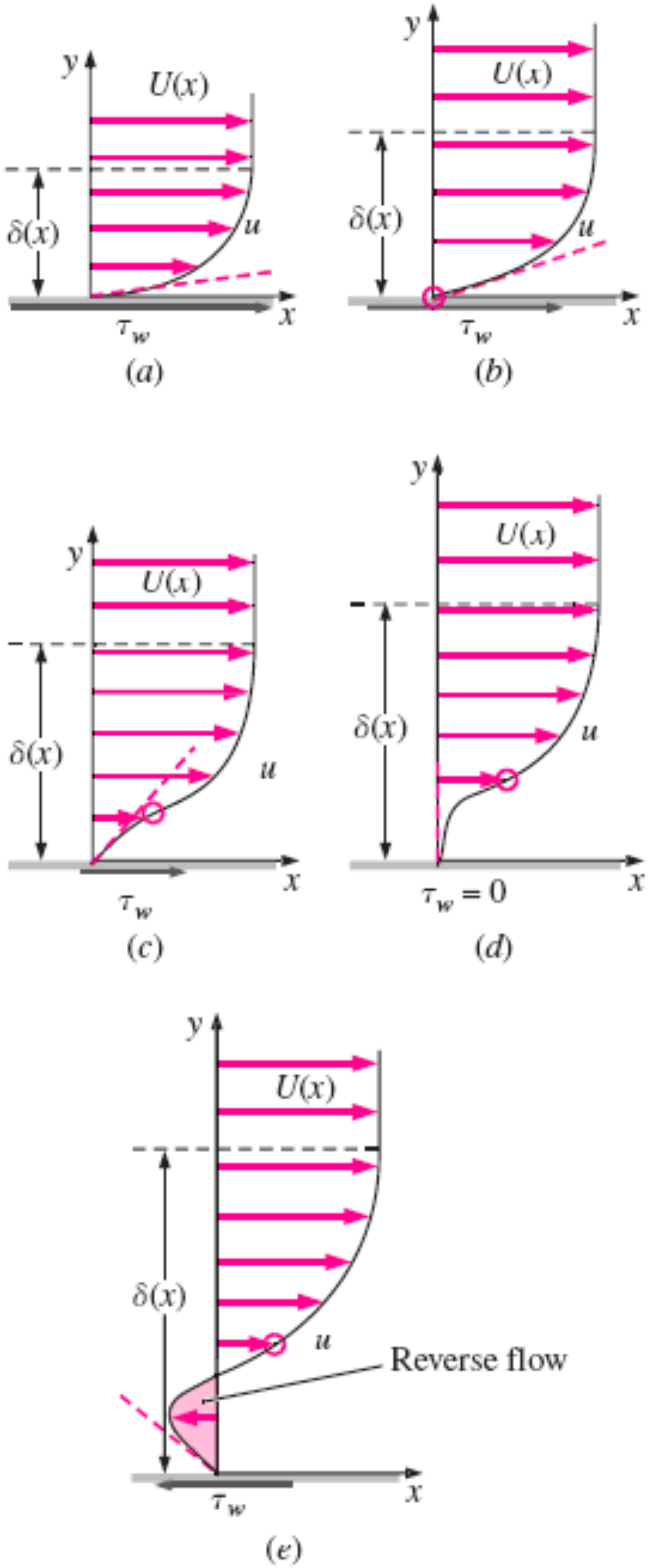


Boundary Layer Procedure

- Possible Limitations
  1. Re is not large enough ⇒ BL may be too thick for thin BL assumption.
  2. $\partial p/\partial y \neq 0$ due to wall curvature $\delta \sim R$
  3. Re too large ⇒ turbulent flow at Re = 1×10^5. BL approximation still valid, but new terms required.
  4. Flow separation



Pressure Gradients



- Shape of the BL is strongly influenced by external pressure gradient:

  (a) favorable ($dP/dx < 0$)

  (b) zero

  (c) mild adverse ($dP/dx > 0$)

  (d) critical adverse ($\tau_w = 0$)

  (e) large adverse with reverse (or separated) flow

Pressure Gradients

- The BL approximation is not valid downstream of a separation point because of reverse flow in the separation bubble.
- Turbulent BL is more resistant to flow separation than laminar BL exposed to the same adverse pressure gradient