22 Nov Mattina

$$sh(x) = \frac{e^{x} - e^{-x}}{2}, \quad ch(x) = \frac{e^{x} + e^{-x}}{2}$$

$$th(x) = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

<u>Lemma</u> $(sh\,x)' = ch\,x, \quad (ch\,x)' = sh\,x$

<u>Dim</u>

$$(ch\,x)' = \left(\frac{e^{x}+e^{-x}}{2}\right)' = \frac{1}{2}\left(e^{x}+e^{-x}\right)' =$$

$$= \frac{d}{dx}\,\frac{e^{x}+e^{-x}}{2} = \frac{1}{2}\,\frac{d}{dx}\left(e^{x}+e^{-x}\right)$$

$$= \frac{1}{2}\left(\frac{d}{dx}e^{x} + \frac{d}{dx}e^{-x}\right) = \frac{1}{2}\left(e^{x} + (e^{y})'(-x)\,(-x)'\right)$$

$$= \frac{1}{2}\left(e^{x} + e^{-x}(-1)\,(x)'\right) = \frac{1}{2}\left(e^{x} - e^{-x}\right) = sh(x)$$

$$(\operatorname{th} x)' = \left(\frac{\operatorname{sh} x}{\operatorname{ch} x}\right)' = \frac{(\operatorname{sh} x)' \operatorname{ch} x - \operatorname{sh} x \, (\operatorname{ch} x)'}{\operatorname{ch}^2(x)} =$$

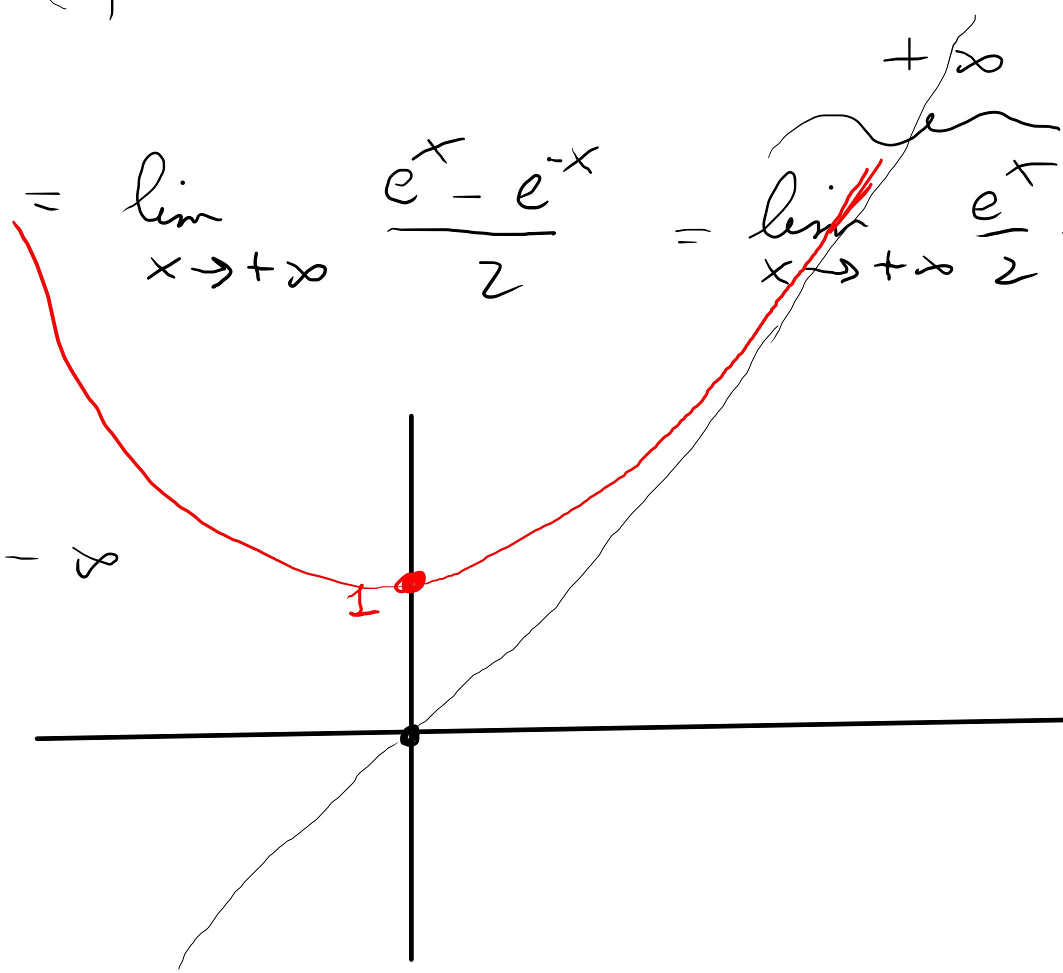
$$= \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{ch}^2(x)} = \frac{1}{\operatorname{ch}^2 x}$$

$$= 1 - \operatorname{th}^2(x)$$

$$\lim_{x \to +\infty} \operatorname{sh}(x) = \lim_{x \to +\infty} \frac{e^x - e^{-x}}{2} = \underbrace{\lim_{x \to +\infty} \frac{e^x}{2}}_{+\infty} - \underbrace{\lim_{x \to +\infty} \frac{e^{-x}}{2}}_{0}$$

$$= +\infty$$

$$\lim_{x \to -\infty} \operatorname{sh}(x) = -\infty$$

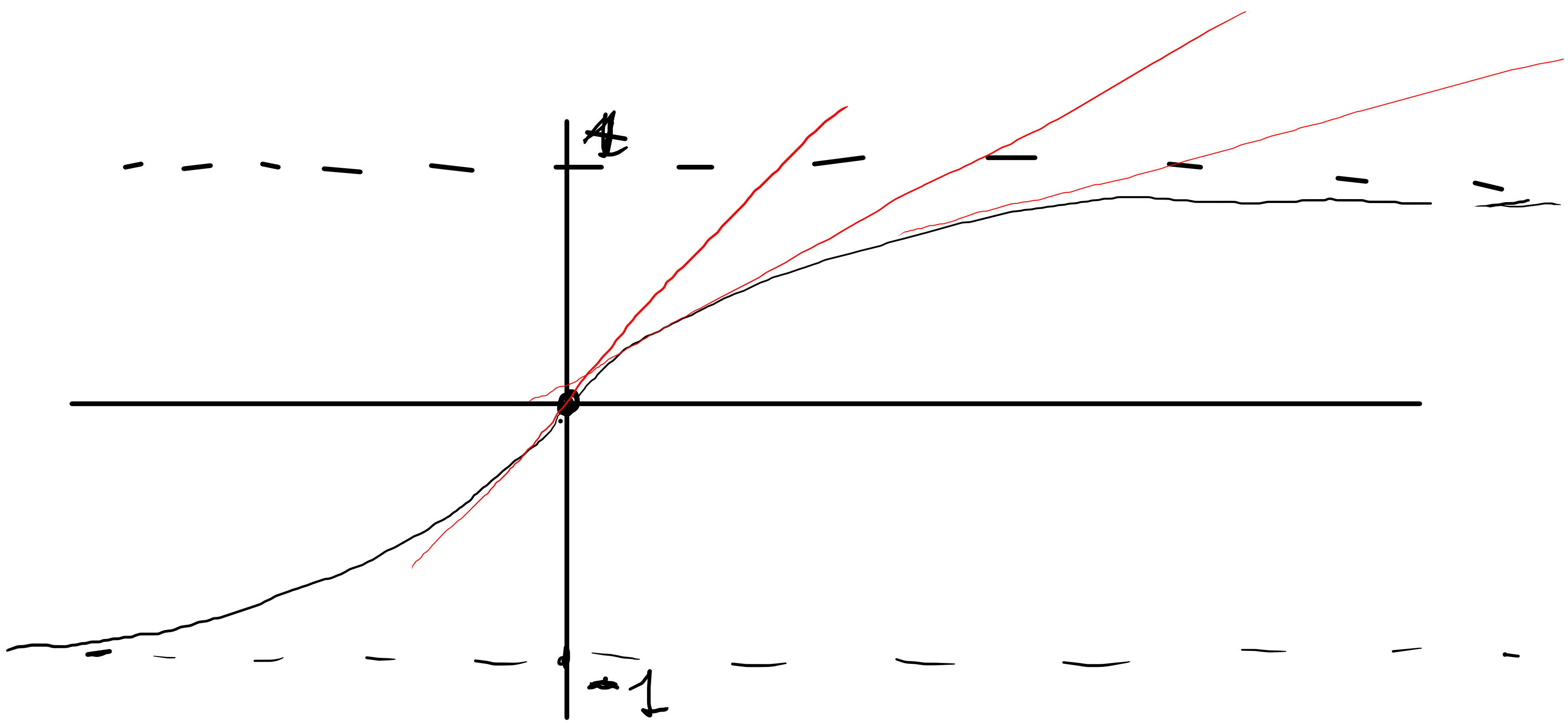


$$\operatorname{ch}(x) = \frac{e^x + e^{-x}}{2} > \frac{e^x - e^{-x}}{2} = \operatorname{sh}(x) \qquad \operatorname{ch}(0) = 1$$

$$\lim_{x \to \infty} \operatorname{ch} x = +\infty$$

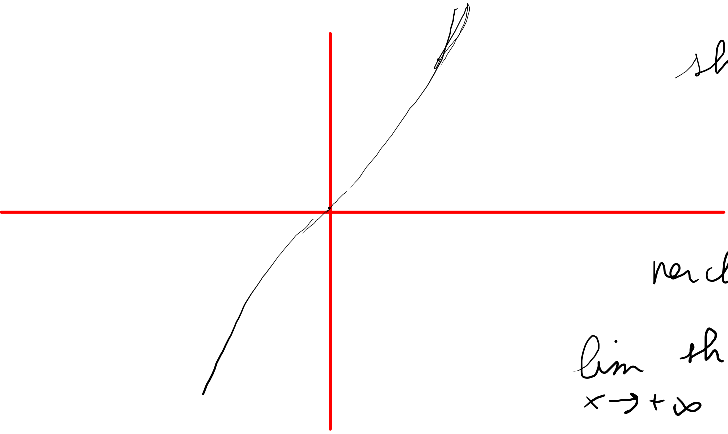
$\mathrm{ch}(x) > \mathrm{sh}(x) \qquad 1 > \dfrac{\mathrm{sh}(x)}{\mathrm{ch}(x)}$

$$-1 < \mathrm{th}\, x = \frac{\mathrm{sh}\, x}{\mathrm{ch}\, x} < 1 \qquad \forall x$$



$$\lim_{x\to+\infty} \mathrm{th}\, x = \lim_{x\to+\infty} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} = \lim_{x\to+\infty} \frac{e^{x}}{e^{x}} = 1$$

$$(\mathrm{th}\, x)' = 1 - \mathrm{th}^2(x)$$



$sh(x) : \mathbb{R} \to \mathbb{R}$

è suriettiva

perché

$$\lim_{x\to+\infty} sh(x) = \sup sh(\mathbb{R}) = +\infty$$

$$\lim_{x\to-\infty} sh(x) = \inf sh(\mathbb{R}) = -\infty$$

$$\left.\begin{array}{l}\sup sh(\mathbb{R}) = +\infty \\ \inf sh(\mathbb{R}) = -\infty\end{array}\right. \Rightarrow sh(\mathbb{R}) = (-\infty, +\infty) = \mathbb{R}$$

~~e^x~~ $sh(x)$ è iniettiva perché strettamente crescente

C'è una funzione inversa.

$$y = sh(x) \qquad y = \frac{e^x - e^{-x}}{2}$$

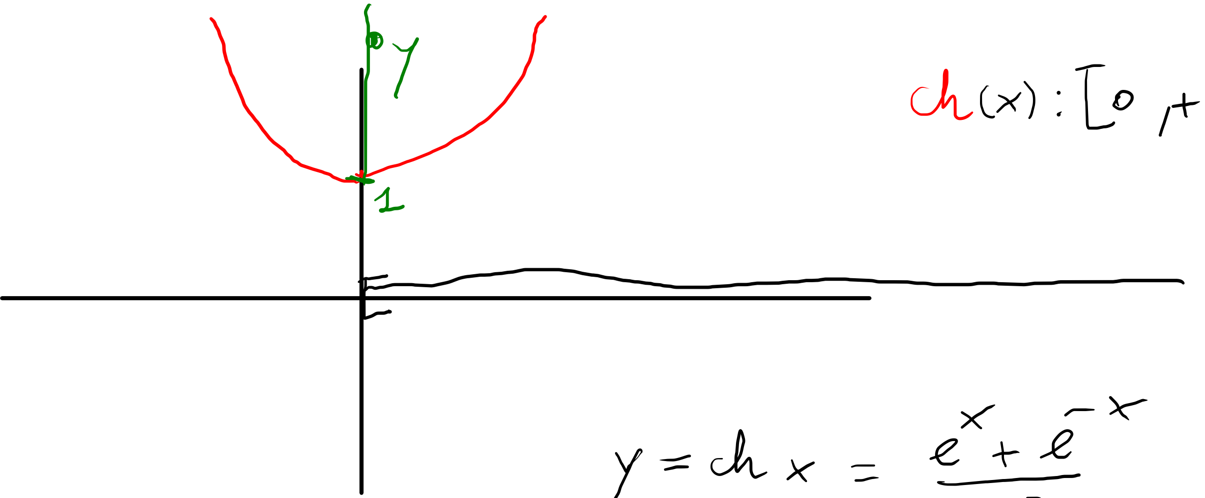
$$2y = e^x - e^{-x} \qquad \cdot e^x$$

$$2y e^x = e^{2x} - 1 \qquad (e^x)^2 - 2y e^x - 1 = 0$$

$$(e^x)_\pm = y \pm \sqrt{y^2+1} \begin{cases} \cancel{y - \sqrt{y^2+1}} < 0 \\ y + \sqrt{y^2+1} \end{cases}$$

$$e^x = (e^x)_+ = y + \sqrt{y^2+1}$$

$$x = \lg\left(y + \sqrt{y^2+1}\right)$$



$$ch(x) : [0, +\infty) \to [1, +\infty)$$

$$y = ch\, x = \frac{e^x + e^{-x}}{2}$$

$$2y = e^x + e^{-x} \qquad \cdot e^x$$

$$2y e^x = e^{2x} + 1 \qquad (e^x)^2 - 2y\, e^x + 1 = 0$$

$$(e^x)_{\pm} = y \pm \sqrt{y^2 - 1}$$

$$\cancel{y - \sqrt{y^2-1} \leq 1} \qquad y \geq 1$$

$$y + \sqrt{y^2-1}$$

Verifichiamo che $y - \sqrt{y^2-1} < 1$ ✓ $\qquad y > 1$

$$0 < y - 1 < \sqrt{y^2-1}$$

$$\Updownarrow$$

$$(y-1)^2 < y^2 - 1$$

$$\cancel{y^2} - 2y + 1 < \cancel{y^2} - 1 \iff 0 < 2y - 2 = 2\underbrace{(y-1)}_{\geq 0}$$

Esercizio Dimostrare l'equivalenza di

1) $\lim_{x\to x_0} f(x)$ esiste

2) $\lim_{x\to x_0^+} f(x)$ e $\lim_{x\to x_0^-} f(x)$ esistono e sono uguali

Dim $1 \Rightarrow 2$

Se 1) è vero allora $\lim_{x\to x_0} f(x) = L \in \overline{\mathbb{R}}$. Qui sia $L \in \mathbb{R}$.

Allora 1) significa che

(3) $\forall \varepsilon>0\ \exists\, \delta_\varepsilon>0$ t.c. $0<|x-x_0|<\delta_\varepsilon$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$

La (3) implica, con lo stesso δ_ε della (3),

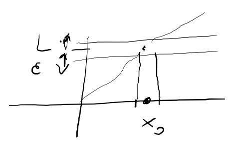
(4) $\forall \varepsilon>0\ \exists\, \delta_\varepsilon>0$ t.c. $0<x-x_0<\delta_\varepsilon$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$

e

(5) $\forall \varepsilon>0\ \exists\, \delta_\varepsilon>0$ t.c. $0<x_0-x<\delta_\varepsilon$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$

Ma (4) $\Rightarrow \lim_{x\to x_0^+} f(x) = L$

(5) $\Rightarrow \lim_{x\to x_0^-} f(x) = L$



Conclusione 1) $\Rightarrow$ 2). Ora dimostriamo (2) $\Rightarrow$ 1)

2) ci garantisce che $\exists\, L\in\overline{\mathbb{R}}$ t.c. $\lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = L$

Consideriamo il caso $L\in\mathbb{R}$

$\lim_{x\to x_0^+} f(x) = L$ e $\lim_{x\to x_0^-} f(x) = L$ implicano

(3') $\forall \varepsilon>0\ \exists\, \delta_\varepsilon^+>0$ t.c. $0<x-x_0<\delta_\varepsilon^+$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$

(4') $\forall \varepsilon>0\ \exists\, \delta_\varepsilon^->0$ t.c. $0<x_0-x<\delta_\varepsilon^-$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$

Se pongo $\boxed{\delta_\varepsilon = \min\{\delta_\varepsilon^+, \delta_\varepsilon^-\}}$ segue che

$0<|x-x_0|<\delta_\varepsilon$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$

Infatti se $x\in X$ e $0<|x-x_0|<\delta_\varepsilon$, due sono i casi

(a) $x-x_0>0$. Allora $0<x-x_0<\delta_\varepsilon \le \delta_\varepsilon^+$
$\overset{(3')}{\Longrightarrow} |f(x)-L|<\varepsilon$

(b) $x-x_0<0$. Allora $0<x_0-x<\delta_\varepsilon \le \delta_\varepsilon^-$
$\overset{(4')}{\Longrightarrow} |f(x)-L|<\varepsilon$

Abbiamo dimostrato che $\forall\varepsilon>0\ \exists\,\delta_\varepsilon\ (=\min\{\delta_\varepsilon^+,\delta_\varepsilon^-\})>0$ t.c. $0<|x-x_0|<\delta_\varepsilon$ e $x\in X \Rightarrow |f(x)-L|<\varepsilon$