$$\mathrm{Tr}\log\Delta_{gh} \underset{\substack{\text{div.}\\\text{part}}}{\approx} -\frac{1}{6}\,\frac{1}{(4\pi)^2}\,C_2(G)\,\delta^{ab}\int\frac{d^dp}{(2\pi)^d}\,\tilde{A}^{\mu a}(p)\,\tilde{A}^{\nu b}(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})\,\frac{1}{2-\omega}$$

$$\mathrm{Tr}\log\Delta^{\mu\nu}_{gauge} \approx \frac{10}{3}\,\frac{1}{(4\pi)^2}\,C_2(G)\,\delta^{ab}\,\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\,\hat{A}^a_\mu(p)\,\hat{A}^b_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\Downarrow$$

$$S_{eff}(A) = \frac{1}{2g^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) + \underbrace{\frac{1}{2}\mathrm{Tr}\log\Delta_{gauge} - \mathrm{Tr}\log\Delta_{gh}}_{\frac{C_2(G)}{(4\pi)^2}\underbrace{\left[\frac{1}{2}\cdot\frac{10}{3} - \left(-\frac{1}{6}\right)\right]}_{11/6}\frac{1}{2-\omega}\int\frac{d^4p}{(2\pi)^4}\tilde{A}^a_\mu(p)\tilde{A}^a_\nu(-p)(p^\mu p^\nu - p^2\delta^{\mu\nu})}$$

$$\frac{1}{4g^2}\int d^4x\,F^a_{\mu\nu}F^{a\,\mu\nu}$$

$$-\frac{1}{2g^2_{bare}}\int\frac{d^4p}{(2\pi)^4}\,\tilde{A}^a_\mu(p)\,\tilde{A}^a_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

RINORMALIZZAZIONE

$$\Downarrow$$

$$-\frac{\mu^{4-2\omega}}{2g^2_{bare}} + \frac{1}{2}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{11}{3}\,\frac{1}{2-\omega} = -\frac{1}{2g_r^2(\mu)}$$

$$\frac{(\mu^{2-\omega})^2}{g^2_{bare}} = \frac{1}{g_r^2} + \frac{11}{3}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega} = \frac{1}{g_r^2}\left(1 + \frac{11}{3}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega}\,g_r^2\right)$$

$$\rightarrow\ g_{bare} = g_r(\mu)\,\mu^{2-\omega}\left(1 - \frac{11}{6}\,\frac{C_2(G)}{(4\pi)^2}\,\frac{1}{2-\omega}\,g_r^2(\mu)\right)$$

$$\beta(g) = -\frac{11}{3}\,\frac{C_2(G)}{16\pi^2}\,g^3$$

β-funct di YM

Contributo fermioni

Se abbiamo un fermione in rep. R di G, abbiamo

$$e^{-S_{eff}(A)} = \int e^{-S(A,c,\bar{c},a)} \mathcal{D}a\mathcal{D}c\mathcal{D}\bar{c} \underbrace{e^{-S_f(\psi,A,a)}\mathcal{D}\psi\mathcal{D}\bar{\psi}}_{\det i\not{D}}$$

$$\not{D} = \gamma^\mu D_\mu$$

$\rightarrow$ contributo a S_{eff} pari a

$$-\log\det(i\not{D}) \qquad \left(\det M = \sqrt{\det(M^2)}\right)$$

$$\det(i\not{D}) = \det{}^{1/2}\left(-\gamma^\mu\gamma^\nu D_\mu D_\nu\right) =$$

$$\frac{1}{2}[D_\mu, D_\nu] = -\frac{i}{2}F_{\mu\nu}$$

$$= \det{}^{1/2}\Big(-\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2\delta^{\mu\nu}\mathbb{1}_4} D_\mu D_\nu - \frac{1}{2}[\gamma^\mu,\gamma^\nu]\overbrace{D_\mu D_\nu}\Big) =$$

$$= \det{}^{1/2}\left(-D^2\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)$$

Se ho N_f fermioni devo elevare q.to det alla N_f

anche su indici spinoriali

$$\Rightarrow -\log\det(i\not{D}) = -\frac{1}{2}\mathrm{Tr}\log\Big(-D^2\mathbb{1}_4 + \underbrace{\frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}_{(\dots)}\Big)$$

N_f compare come fattore

$$= -\frac{1}{2}\mathrm{Tr}\log\Big(-\partial^2\mathbb{1}_4 + \overbrace{(\Delta_1+\Delta_2)\mathbb{1}_4 + \frac{i}{4}[\gamma^\mu,\gamma^\nu]F_{\mu\nu}}^{(\dots)}\Big) =$$

espando log

$$= \underbrace{-\frac{1}{2}\mathrm{Tr}\cancel{\log}(-\partial^2\mathbb{1}_4)}_{\text{cost. che trascuriamo}} - \frac{1}{2}\mathrm{Tr}\log\left(1 + (-\partial^2)^{-1}(\dots)\right) \simeq$$

$$= -\frac{1}{2}\mathrm{Tr}\left((-\partial^2)^{-1}(\dots)\right) - \frac{1}{2}\left(-\frac{1}{2}\mathrm{Tr}\left[\left((-\partial^2)^{-1}(\dots)\right)^2\right]\right)$$

$\mathrm{tr}\, t^a_R = 0 \Rightarrow$

$\Rightarrow F_{\mu\nu}$ non compare

termini misti $F_{\mu\nu}\Delta_1$ fanno zero perché $\mathrm{tr}[\gamma^\mu,\gamma^\nu] = 0$ (cicl. tr.)

termini Δ_2 e Δ_1 ricostr. $\mathrm{Tr}\log(-D^2)\mathbb{1}_4$

$$= \underbrace{-\overset{N_f}{\frac{1}{2}}\mathrm{Tr}\log(-D^2\mathbb{1}_4)}_{-2\mathrm{Tr}\log(-D^2)} - \overset{N_f}{\frac{1}{2}}\Big\{\underbrace{-\frac{1}{2}\left(\frac{i}{4}\right)^2\mathrm{Tr}\left((-\partial^2)^{-1}F^a_{\mu\nu}t^a_R[\gamma^\mu,\gamma^\nu](-\partial^2)^{-1}F^b_{\rho\sigma}t^b_R[\gamma^\rho,\gamma^\sigma]\right)}_{F_{\mu\nu}\text{ - terms}}\Big\}$$

$$\{\cdots\} = -\frac{1}{2}\left(-\frac{1}{16}\right)\int d^dy\, d^dx \langle y|(-\partial^2)^{-1}|x\rangle\langle x|(-\partial^2)^{-1}|y\rangle F^a_{\mu\nu}(x) F^b_{\rho\sigma}(y)\,\cdot$$

$$\cdot\, \mathrm{tr}\, t^a_R t^b_R \;\underbrace{\mathrm{tr}_f\left([\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\right)}_{-16\cdot 2\,\delta^{\mu\rho}\delta^{\nu\sigma}}$$

$$\mathrm{tr}_f\left([\gamma^\mu,\gamma^\nu][\gamma^\rho,\gamma^\sigma]\right) = \mathrm{tr}_f\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + (\mu\leftrightarrow\nu, \rho\leftrightarrow\sigma)$$

$$= 4(\delta^{\mu\nu}\delta^{\rho\sigma} \underbrace{- \delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho}}_{\text{antisim. in } \mu\nu \text{ e } \rho\sigma\text{, ma sim. in } \mu\nu\ \rho\sigma \text{ simultaneamente}}) - (\mu\leftrightarrow\nu) - (\rho\leftrightarrow\sigma) + (\mu\leftrightarrow\nu, \rho\leftrightarrow\sigma)$$

$$= 4\,\delta^{\mu\nu}\delta^{\rho\sigma}(1-1-1+1) + 4\cdot 4\,(-\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho})$$

$$= 16\,(-\delta^{\mu\rho}\delta^{\nu\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho})$$

$$= -\int d^dy\, d^dx \langle y|(-\partial^2)^{-1}|x\rangle\langle x|(-\partial^2)^{-1}|y\rangle F^a_{\mu\nu}(x) F^{b\,\mu\nu}(y)\, \underbrace{\mathrm{tr}\, t^a_R t^b_R}_{c(R)\,\delta^{ab}}$$

Da calcolo di $\Delta^{\mu\nu}_{\text{gauge}}$, sappiamo che

$$-\cancel{2}\int d^dy\, d^dx \langle y|(-\partial^2)^{-1}|x\rangle\langle x|(-\partial^2)^{-1}|y\rangle F^{\mu\nu a}(x) F^b_{\mu\nu}(y)\, \cancel{\mathrm{tr}(t^a_{Ad} t^b_{Ad})}$$

$$= \overset{2}{\cancel{4}}\int \frac{d^dp}{(2\pi)^d} \tilde{A}^a_\mu(p)\tilde{A}^b_\nu(-p)\, \cancel{\mathrm{tr}(t^a t^b)}\,(p^\mu p^\nu - p^2\delta^{\mu\nu}) \int_0^1 d\xi\, \frac{\Gamma(2-d/2)}{(\xi(1-\xi)p^2)^{2-\frac{d}{2}}(4\pi)^{d/2}}$$

$$\underset{\text{div.}}{\sim} \frac{2\cancel{4}\,\cancel{c(Ad)\,\delta^{ab}}}{(4\pi)^2}\,\frac{1}{2-\omega}\int \frac{d^dp}{(2\pi)^d} \hat{A}^a_\mu(p)\hat{A}^b_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\underset{\text{div.}}{\sim} 2\,c(R)\,\frac{1}{(4\pi)^2}\,\frac{1}{2-\omega}\int \frac{d^dp}{(2\pi)^d} \hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)\,(p^\mu p^\nu - p^2\delta^{\mu\nu})$$

$$\Rightarrow -\mathrm{Tr}\log i\not{D} = -2\,\mathrm{Tr}\log(-D^2) - \frac{1}{2}\{\cdots\} =$$

$$= -\cancel{2}\left(-\frac{1}{\cancel{6}\,3}\right)\frac{c(R)}{(4\pi)^2}\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$

$$-\frac{1}{\cancel{2}}\left(\overset{1}{\cancel{2}}\right)\frac{c(R)}{(4\pi)^2}\frac{1}{2-\omega}\int\frac{d^dp}{(2\pi)^d}\hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$

$$= -\frac{2}{3}\frac{c(R)}{(4\pi)^2}\frac{1}{2-\omega}\overset{N_f}{\vee}\int\frac{d^dp}{(2\pi)^d}\hat{A}^a_\mu(p)\hat{A}^a_\nu(-p)\left(p^\mu p^\nu - p^2\delta^{\mu\nu}\right)$$

Quindi ora

$$-\frac{\mu^{4-2\omega}}{2g^2_{bare}} + \frac{1}{2}\left(\frac{C_2(G)}{(4\pi)^2}\frac{11}{3}\frac{1}{2-\omega} - \frac{4}{3}\frac{c(R)}{(4\pi)^2}\frac{N_f}{2-\omega}\right) = -\frac{1}{2g_r^2(\mu)}$$

$$\Rightarrow \boxed{\beta(g) = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}C_2(G) - \frac{4}{3}N_f\, c(R)\right)}$$

Funzione β per teorie non-abeliane con gruppo di gauge G e accoppiate a N_f fermioni di Dirac in rep. R di G.

Nota: se introducessimo anche N_s SCALARI in rep R_s di G, con azione

$$S_s = \int d^4x \left(D_\mu \phi^\dagger D^\mu \phi - V(\phi) \right)$$

anch'essi contribuirebbero alla funz. β che diventerebbe

$$\beta(g) = \beta_0 g^3 + \dots$$

dove $$\beta_0 = -\frac{1}{(4\pi)^2}\left[\frac{11}{3} c_2(G) - \frac{4}{3} N_f c(R) - \frac{1}{3} N_s\, c(R_s) \right]$$

Questo si può capire ricordando che il contributo dei ghost (scalari in rep. Adj, ma a valori nei numeri di Grassmann) danno contributo $+\frac{1}{3}c(\text{Adj})$. Rispetto ai ghost, quindi, cambia segno overall (integrale Gaussiano da inverso del det), $c(\text{Adj}) \to c(R_s)$ e fattore N_s, per dare conto della possibile molteplicità dei campi scalari.

Nota 2: Il conto per la funzione β è stato fatto per materia MASSLESS. Infatti nel det compariva solo $i\not{D}$, senza termine di massa (e per scalari solo D^2).