Def 23 Nov E t.v.s. is separable if it contains a $X \subseteq E$

X dense in E

X countable.

Ex $C^0([a,b]) \supseteq \mathbb{R}[x]$

$\mathbb{Q}[x]$

Lemma E B-space. If E' is separable also E is separable

Remark E sep $\not\Rightarrow$ E' sep

$$\underbrace{E = L^1(-1,1)}_{\text{sep}}, \qquad \underbrace{E' = L^\infty(-1,1)}_{\text{not separab}}$$

Pf Let $\{f_n\}$ be dense in E'

For any f_n $\exists$ $x_n \in E$

$\|x_n\|_E = 1$ st $f_n(x_n) \geq \frac{1}{2} \|f_n\|_{E'}$

$\{x_n\}_{n \in \mathbb{N}} \subseteq E$

$L = \overline{\text{Span}\{x_n : n \in \mathbb{N}\}} \subseteq E$

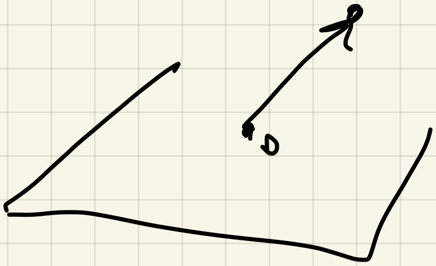
L is separable

If E not separable $L \subsetneq E$

$\Rightarrow \exists f \in E' \setminus \{0\}$ st

$f|_L = 0$ so in particular

$\boxed{f(x_n) = 0}$



$\exists f_{n_k} \xrightarrow{k \to +\infty} f$ in E'

$\|f_{n_k} - f\|_{E'} \geq f_{n_k}(x_{n_k}) - \overset{= 0}{\boxed{f(x_{n_k})}}$

$\geq \frac{1}{2} \|f_{n_k}\|_{E'}$

$$\begin{tikzcd}
\underbrace{\|f_{n_k}-f\|_{E'}} \arrow[d, "k\to+\infty"] & \geqslant & \frac{1}{2}\underbrace{\|f_{n_k}\|_{E'}} \arrow[d] \\
0 & \geqslant & \frac{1}{2}\|f\|_{E'}
\end{tikzcd}$$

$\Longrightarrow \quad f \equiv 0$. A contradiction.

$L = E$.

Exercise $\quad E$ reflexive and separable $\overset{\Longleftarrow}{\Longleftrightarrow}$

E' reflexive and separable

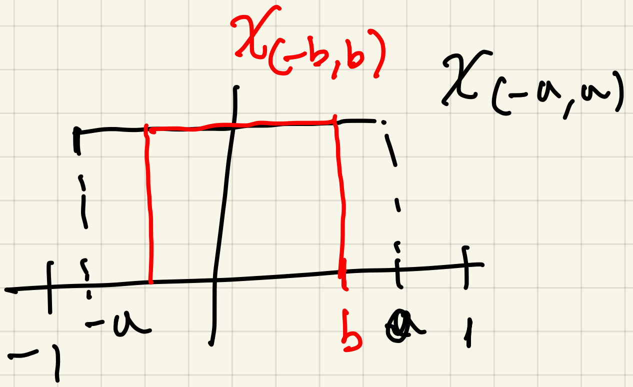
$\Longrightarrow \quad E = E''$

Lemma $\quad L^\infty(-1,1)$ is not separable

$a \in (0,1)$

$I_a = (-a,a)$

$\chi_{(-a,a)}$



$$\boxed{\begin{aligned} & D_{L^\infty(-1,1)}\left(\chi_{(-a,a)}, \tfrac{1}{2}\right) \\ & \cap\ D_{L^\infty(-1,1)}\left(\chi_{(-b,b)}, \tfrac{1}{2}\right) = \phi \qquad a \neq b \end{aligned}}$$

because if it was not empty

$$\| f - \chi_{(-a,a)} \|_\infty < \frac{1}{2}$$

$$\chi_{(-b,b)} \quad < 1/2$$

$$\Rightarrow \| \underbrace{\chi_{(-a,a)} - \chi_{(-b,b)}} \|_\infty < 1$$

$$0 < b < a \qquad \| \chi_{(-a,-b] \cup [b,a)} \|_\infty = 1$$

$$\left\{ D_{L^\infty(-1,1)}\left(\chi_{(-a,a)}, \tfrac{1}{2} \right) \right\}_{a \in (0,1)}$$

If $L^\infty(-1,1)$ was separable, there would be X contable and dense in $L^\infty(-1,1)$

$$X \cap \underset{\lightning}{D}\left(\chi_{(-a,a)}, \tfrac{1}{2} \right) \neq \phi \quad \forall a$$

Impossible $\qquad x_a \in X \quad \forall\ a \in (0,1)$

$$Y = \{ x_a , a \in (0,1) \}$$

$$\begin{tikzcd} Y \arrow[r] & (0,1) \\ x_a \arrow[r] & a \end{tikzcd}$$

Example $L^\infty(X, \mathbb{R})$

$$\overline{\text{Span}\{ \chi_E : E \text{ measurable subset of } X \}}$$

$$= L^\infty(X, \mathbb{R})$$

Lemma E Banach and reflexive.

and if $\{x_n\}$ is a bounded sequence then $\exists$ $\{x_{n_k}\}$ weakly $\sigma(E, E')$ convergent in E.

Pf $F = \overline{\text{Span}\{x_n : n \in \mathbb{N}\}}$

$F \overset{\text{closed}}{\subseteq} E \Rightarrow$ (F is reflexive and separable) $\Longleftrightarrow$ (F^* is ref and sep)

Then in the $\sigma(F, F')$ top

Every bounded subspace of F is metrizable.

So $\exists$ $\{x_{n_k}\}$ $x_{n_k} \longrightarrow x$ in F

for the $\sigma(F, F')$

But $(F, \sigma(F, F'))$

$(F, \sigma(E, E')|_F)$

are the same.

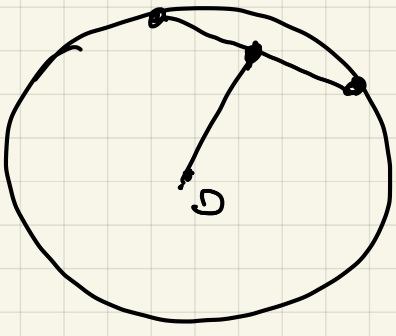
Uniformly convex spaces

Def E B space is unif convex.

if $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ s t

if $x, y \in \overline{D_E(0,1)}$ and if

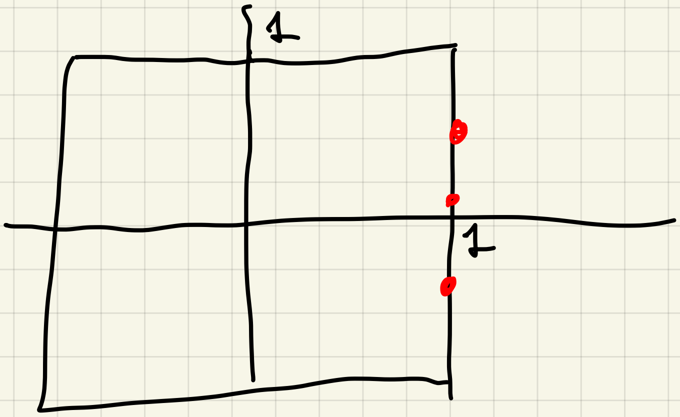
$\|x-y\|_E > \varepsilon$ then $\left\| \frac{x+y}{2} \right\|_E < 1 - \delta$



$\mathbb{R}^2$

$\sqrt{x_1^2 + x_2^2}$

$\mathbb{R}^2$ $\|x\| = \sup\{|x_1|, |x_2|\}$



$$\|x\|^2 + \|y\|^2 = \frac{\|x-y\|^2}{2} + \frac{\|x+y\|^2}{2}$$

Th E B-spac uniformly convex $\Longrightarrow$ E reflexive

Pf We want to show that

$$J\,\overline{D_E(0,1)} = \overline{D_{E''}(0,1)}$$

Let $\|\xi\|_{E''} = 1$, we want to show that

$$\xi \in \underbrace{J\,\overline{D_E(0,1)}}_{\text{is closed in } E''}$$

It is enough to show that $\forall \varepsilon > 0$

$\exists \quad x \in \overline{D_E(0,1)}$ s.t.

$$\|\xi - Jx\|_{E''} \leq \varepsilon \quad , \quad \delta$$

$\exists \; f \in E'$ s.t. $\|f\|_{E'} = 1$

$$\langle \xi, f \rangle_{E'' \times E'} > 1 - \frac{\delta}{2}$$

$$V = \{ \eta \in E'' : | \langle \eta - \xi, f \rangle | < \tfrac{\delta}{2} \}$$

nbhd of ξ for $\sigma(E'', E')$ top.

$$V \ni \xi \in \overline{D_{E''}(0,1)}$$

$J \overline{D_E(0,1)}$ is dense in $\overline{D_{E''}(0,1)}$

for $\sigma(E'', E')$ top.

$\Longrightarrow \quad \exists \quad x \in \overline{D_E(0,1)}$ s.t. $Jx \in V$

$Jx + \varepsilon \overline{D_{E''}(0,1)}$ in E''

If ξ does not belongs, let us call W the complement in E'' of $Jx + \varepsilon \overline{D_{E''}(0,1)}$

So $\xi \in W$. W is open in E''

for the $\sigma(E'', E')$

$W \cap V$ is nonempty closed

$$W \cap V \cap \overline{D_{E''}(0,1)} \neq \phi$$

Since $J\,\overline{D_E(0,1)}$ is dense in $\overline{D_{E''}(0,1)}$

$\exists \quad y \in \overline{D_E(0,1)}$ st

$Jy \in W \cap V \qquad y \neq x$

$Jx \notin W$

$Jx, \; Jy \in V$

$$-\frac{\delta}{2} < -\langle x, f\rangle_{E\times E'} + \langle \xi, f\rangle_{E''\times E'} < \frac{\delta}{2}$$

$$-\frac{\delta}{2} < -\langle y, f\rangle_{E\times E'} + \langle \xi, f\rangle_{E''\times E'} < \frac{\delta}{2}$$

$$2-\delta < 2\langle \xi, f\rangle_{E''\times E'} < \langle x+y, f\rangle_{E\times E'} + \delta$$

$$\le \|x+y\|_E + \delta$$

$$2-\delta < \|x+y\|_E + \delta$$

$$\boxed{1-\frac{\delta}{2}} < \left\|\frac{x+y}{2}\right\|_E + \frac{\delta}{2} \boxed{< 1-\delta+\frac{\delta}{2}}$$

$$\|Jy - Jx\|_{E''} > \varepsilon \iff \|x-y\|_E > \varepsilon$$

$$\Rightarrow \quad \left\|\frac{x+y}{2}\right\|_E < 1-\delta$$

$1-\frac{\delta}{2} < 1-\frac{\delta}{2}$ impossible

$$1 \le p < +\infty$$

$L^p(X, d\mu) = \{ f$ meas : $|f|^p \in L^1(X, d\mu)\}$

$L^\infty(X, d\mu) = \{ f$ meas st $|f(x)| \le C < +\infty$ a. e p

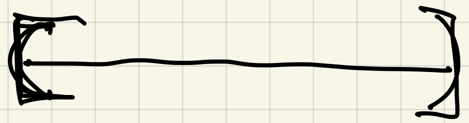
$$1 \le p < \infty$$

$$\|f\|_{L^p} = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}}$$

$$\|f\|_{L^\infty} = \sup \{ c \ge 0 : \mu(\{x : |f(x)| \ge c\}) > 0$$

Remark If $f_n \longrightarrow f$ in $L^p(0,1)$ $1 \le p < +\infty$ then in general we do not have $f_n(x) \longrightarrow f(x)$ pointwise.

Example of sequence where $f_n \to 0$ in $L^p(0,1)$ but $f_n(x) \not\to 0$ $\forall x \in (0,1)$



$\forall n \in \mathbb{N}$

$\bigcup_{j=1}^{n} \left[\frac{j-1}{n}, \frac{j}{n} \right]$

Let $\{I_n\}$ be a sequence of ~~of~~ intervals

$$\chi_{I_n} \longrightarrow 0 \quad \text{in } L^p$$

$$\|\chi_{I_n}\|_{L^p} = |I_n|^{\frac{1}{p}} \xrightarrow{n\to\infty} 0$$

$\forall x \in (0,1)$

but pointwise $\{\chi_{I_n}(x)\}$

oscillates between $\{0, 1\}$