25 Nov

<u>Thm</u> $L^p(X, d\mu)$ $\quad 1 \le p \le \infty$ are Banach.

<u>Pf</u> $\quad p < +\infty$ Cauchy sequence $\{f_n\}$

We can assume we have a subsequence $\{f_{n_k}\}$ st $\quad \|f_{n_k} - f_{n_\ell}\|_{L^p} < 2^{-k} \quad \forall\ \ell \ge k.$

$$f_{n_k} = f_{n_1} + \sum_{\ell=1}^{k-1} \left(f_{n_{\ell+1}} - f_{n_\ell} \right)$$

$$f_{n_1} + \sum_{\ell=1}^{\infty} \left(f_{n_{\ell+1}} - f_{n_\ell} \right)$$

$$|f_{n_1}| + \sum_{\ell=1}^{\infty} | f_{n_{\ell+1}} - f_{n_\ell} |$$

$$g_k(x) = |f_{n_1}| + \sum_{\ell=1}^{k-1} | f_{n_{\ell+1}} - f_{n_\ell} |$$

$\{g_k\}$ positive and increasing

$$\|g_k\|_{L^p} \le \|f_{n_1}\|_{L^p} + \sum_{\ell=1}^{k-1} 2^{-\ell} \le \|f_{n_1}\|_{L^p} + 1 \quad \forall k$$

$g_k^p(x)$ is an increasing sequence

$$g(x) = \lim_{k\to+\infty} g_k(x)$$

$$g^p(x) = \lim_{k\to+\infty} g_k^p(x) \quad \text{a.e.}$$

$$\int g^p(x)\,dx = \lim_{k\to+\infty} \int g_k^p(x)\,dx < \infty$$

$g(x) < +\infty$ a.e.

and $g \in L^p$

a.e.

$$f(x) = f_{n_1}(x) + \sum_{\ell=1}^{\infty} \left(f_{n_{\ell+1}}(x) - f_{n_\ell}(x) \right)$$

$\ell < k$

$$|f_{n_\ell}(x) - f_{n_k}(x)| \le \sum_{j=\ell}^{k-1} |f_{n_{j+1}}(x) - f_{n_j}(x)|$$

$$\le \sum_{j=\ell}^{+\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| = g(x) - g_\ell(x)$$

$$\le g(x)$$

$$g_k(x) = |f_{n_1}| + \sum_{j=1}^{k-1} |f_{n_{j+1}} - f_{n_j}|$$

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$$g(x) = |f_{n_1}| + \sum_{j=1} |f_{n_{j+1}} - f_{n_j}|$$

$$|f_{n_\ell}(x) - f_{n_k}(x)|^p \le g^p(x) \qquad k \to +\infty$$

$$|f_{n_\ell}(x) - f(x)|^p \le g^p(x)$$

By dominated convergence

is in L^1 with

$$\lim_{k\to+\infty} \|f_{n_\ell} - f_{n_k}\|_{L^p} = \|f_{n_\ell} - f\|_{L^p}$$

$$\cancel{\|f_{n_\ell} - f} \implies f \in L^p$$

$$|f_{n_\ell}(x) - f(x)|^p \le g^p(x)$$

$$\lim_{\cancel{k}\to+\infty} \int |f_{n_\ell} - f|^p =$$

$$= \int \underbrace{\lim_{\ell\to+\infty} |f_{n_\ell} - f|^p}_{0}$$

$f_{n_\ell} \to f$ in L^p

$\overset{\text{olw}}{\Longrightarrow} \quad f_n \longrightarrow f$

Teor $1 < p < \infty$ $\quad L^p$ is reflexive

Teor $2 \leq p < \infty$ Unif convex

Pf $$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \leq \frac{1}{2}\left(\|f\|_{L^p}^p + \|g\|_{L^p}^p \right)$$

$\|f\|_{L^p} \leq 1$, $\|g\|_{L^p} \leq 1$ and

$\|f-g\|_{L^p} > \varepsilon$ $\qquad -\|f-g\|_{L^p}^p < -\varepsilon^p$

$$\left\| \frac{f+g}{2} \right\|_p^p \leq 1 - \frac{\varepsilon^p}{2^p} =$$

$$\left\| \frac{f+g}{2} \right\|_p \leq \left(1 - \frac{\varepsilon^p}{2^p} \right)^{\frac{1}{p}} + 1 - 1$$

$$= 1 - \underbrace{\left(1 - \left(1 - \frac{\varepsilon^p}{2^p} \right)^{\frac{1}{p}} \right)}_{\delta}$$

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \leq \frac{1}{2}\left(\|f\|_{L^p}^p + \|g\|_{L^p}^p \right)$$

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \frac{1}{2}\left(|a|^p + |b|^p\right)$$

I claim that

(1) $$\alpha^p + \beta^p \le (\alpha^2+\beta^2)^{\frac{p}{2}} \qquad 2 \le p < +\infty$$

If I assume (1) $\alpha = \left|\frac{a+b}{2}\right|$, $\beta = \left|\frac{a-b}{2}\right|$

$$\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \le \left(\left|\frac{a+b}{2}\right|^2 + \left|\frac{a-b}{2}\right|^2\right)^{\frac{p}{2}}$$

$$= \left(\frac{|a|^2}{2} + \frac{|b|^2}{2}\right)^{\frac{p}{2}} \le \frac{1}{2}\left(|a|^2\right)^{\frac{p}{2}} + \frac{1}{2}\left(|b|^2\right)^{\frac{p}{2}}$$

$$= \frac{1}{2}|a|^p + \frac{1}{2}|b|^p$$

$t \to t^{\frac{p}{2}}$ $\qquad \frac{p}{2} \ge 1$

(1) $$\alpha^p + \beta^p \le (\alpha^2+\beta^2)^{\frac{p}{2}} \qquad 2 \le p < +\infty$$

I claim it follows from

(2) $$a^q + b^q \le (a+b)^q \quad \text{for any} \quad 1 \le q < \infty$$

$$q = \frac{p}{2} \qquad a = \alpha^2, \quad b = \beta^2$$

$$(\alpha^2)^{\frac{p}{2}} + (\beta^2)^{\frac{p}{2}} \le (\alpha^2+\beta^2)^{\cancel{q}\frac{p}{2}}$$

$$\left(\frac{a}{a+b}\right)^q + \left(\frac{b}{a+b}\right)^q \leq \frac{a}{a+b} + \frac{b}{a+b} = 1$$

Thm L^p $1<p<2$ is reflexive

Pf For $1<p<\infty$ and $f \in L^p$

$\exists$ $Tf \in (L^{p'})'$

$$\langle Tf, g \rangle_{(L^{p'})' \times L^{p'}} = \int f g$$

$$\|Tf\|_{(L^{p'})'} \leq \|f\|_{L^p}$$

$$g(x) = |f(x)|^{p-2} f(x) \qquad \left(\left(f^{p-1}\right)\right)$$

$g \in L^{p'}$ $\qquad p' = \frac{p}{p-1}$

$$\int |g|^{\frac{p}{p-1}} = \int \left(|f|^{\cancel{p-1}}\right)^{\frac{p}{\cancel{p-1}}} = \int |f|^p$$

$$\langle Tf, |f|^{p-2} f \rangle = \int |f|^p \, d\mu = \|f\|_{L^p}^p$$

$$\leq \|Tf\|_{(L^{p'})'} \, \| |f|^{p-1} \|_{L^{\frac{p}{p-1}}}$$

$$= \|Tf\|_{(L^{p'})'} \quad \|f\|^{p-1}_{L^{\frac{p}{\cancel{p-1}}\cancel{(p-1)}}}$$

$$|f^\alpha|_{L^q} = |f|^\alpha_{L^{q\alpha}}$$

$$\|Tf\|_{(L^{p'})'} \quad \cancel{\|f\|^{p-1}_{L^{\frac{p}{\cancel{p-1}}\cancel{(p-1)}}}} \geq \|f\|^{\cancel{p}\ 1}_{L^p}$$

$$\|Tf\|_{(L^{p'})'} = \|f\|_{L^p}$$

$T : L^p \longrightarrow (L^{p'})'$ is an isometry $\quad \forall \ 1 < p < \infty$, $\quad p' = \frac{p}{p-1}$

TL^p is closed subspace of $(L^{p'})'$

$1 < p < 2 \Rightarrow 2 < p' < +\infty$

$\Rightarrow L^{p'}$ is reflexive $\Leftrightarrow (L^{p'})'$ reflexive

Also TL^p is reflexive

$$\begin{tikzcd} T : L^p \arrow[r] & TL^p \arrow[l, bend left] \end{tikzcd}$$

L^p is reflexive.

<u>Teor</u> $\quad 1 < p < \infty \qquad \phi \in (L^p)'$

$\exists \quad u \in L^{p'} \qquad \phi(f) = \int u f$

$\forall f \in L^p$

<u>Pf</u> $\quad T : L^{p'} \longrightarrow (L^p)'$

is an isometry between $L^{p'}$ and $TL^{p'}$

$TL^{p'}$ is a closed subspace of $(L^p)'$

Suppose $\quad TL^{p'} \subsetneq (L^p)'$

$\exists \quad h \in L^p \qquad$ s.t. $\quad h \neq 0$

$\langle Jh, Tu \rangle_{(L^p)'' \times (L^p)'} = 0 \qquad \forall u \in L^{p'}$

$= \langle h, Tu \rangle_{L^p \times (L^p)'} = \int u h = 0$

$\forall u \in L^{p'}$, in particular

for $\quad u = |h|^{p-2} h$

$$0 = \int u h = \int |h|^p \Rightarrow h = 0$$

Theorem $\quad \phi \in \left(L^1(X) \right)'$, $\quad X$ σ- finite

Then $\exists \quad u \in L^\infty(X)$ st

$$\phi(f) = \int u f \, d\mu \quad \forall f \in L^1(X)$$

$$X = \bigcup_{n=1}^{\infty} X_n$$ ~~wit~~ ~~exx~~ with $\mu(X_n) < +\infty$

Pf $\quad X = \bigcup_{n=1}^{\infty} X_n \qquad X_n \subsetneq X_{n+1}$

It is possible to find $w \in L^2(X)$

s.t. $\quad \forall n \ \exists \ C_n > 0 \quad$ s.t.

$$w(x) \geq C_n > 0 \quad \forall x \in X_n.$$

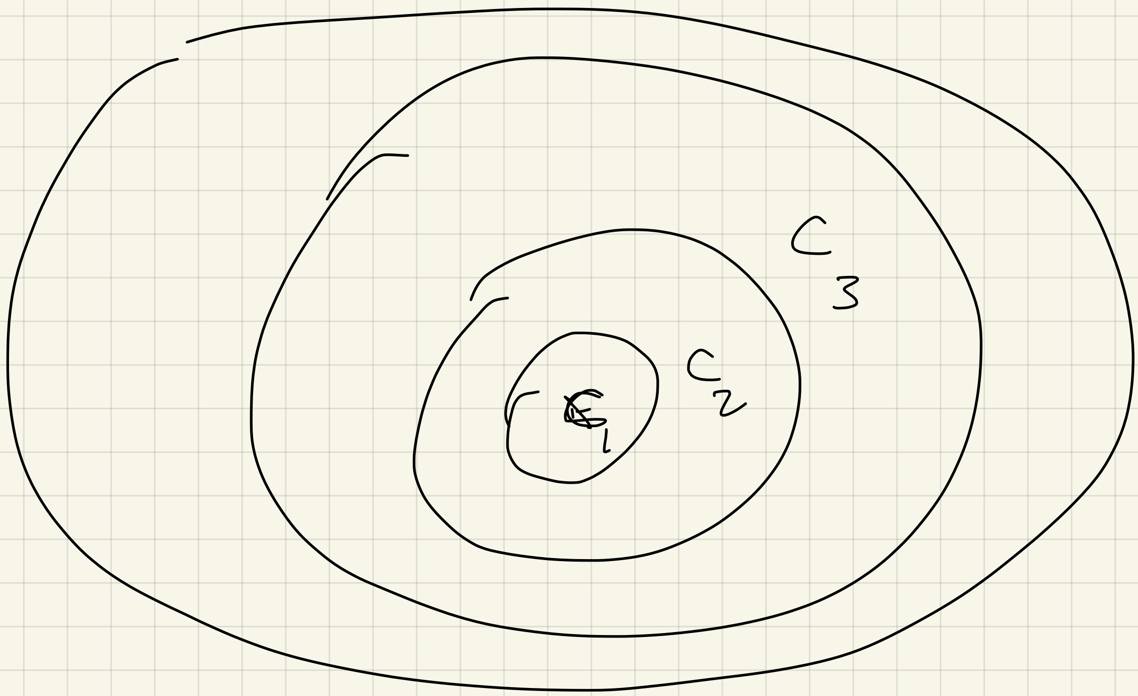
Just use a sequence $\{ c_n \}$ st

$c_n > 0$

$$\sum_{n=1}^{\infty} c_n^2 < +\infty$$

$$w(x) = c_1 \qquad \text{in } X_1$$

$$w(x) = c_n \qquad \text{in } X_n \setminus X_{n-1}$$



$$\int |w|^2 = \cancel{\Sigma} \int_{X_1} c_1^2 + \sum_{n=2}^{+\infty} \int_{X_n \setminus X_{n-1}} c_n^2$$

$$\le c_1^2 \, \mu(X_1) + \sum_{n=2}^{+\infty} c_n^2 \, \mu(X_n) < +\infty$$

$\phi \in (L^1)'$ $\qquad \exists\, g \in L^2$

$f \in L^2 \longrightarrow \langle \phi, f w \rangle = \int f g$

$u = \frac{g}{w}$. We want to show

$\|u\|_{L^\infty} = \|\phi\|_{(L^1)'}$

$\|u\|_{L^\infty} \leq \|\phi\|_{(L^1)'}$

By contradiction suppose $C > \|\phi\|_{(L^1)'}$

$A_\pm = \{ x : \pm u(x) > C \}$

$|A_+| = 0$. Suppose $|A_+| > 0$

$\exists\, n \quad |A_+ \cap X_n| > 0$

$$C \int_{A_+ \cap X_n} w < \int_{A_+ \cap X_n} w u = \int_{A_+ \cap X_n} g$$

$$= \int g \, \chi_{A_+ \cap X_n} = \langle \phi, \chi_{A_+ \cap X_n} w \rangle$$

$$\leq \|\phi\|_{(\cancel{L}^1)'} \int_{A_+ \cap X_n} w$$

$C < \|\phi\|_{(L^1)'}$ a contradiction

$\Rightarrow \quad |A_+| = 0$

$\|u\|_{L^\infty} \leq \|\phi\|_{(L^1)'}$ $\qquad \|u\|_{L^\infty} \geq \|\phi\|_{(L^1)'}$

$L^1 \cap L^2$

$\langle \phi, f \rangle = \int f \, u \qquad \forall f \in L^1$

$\langle \phi, \chi_{X_n} f \rangle = \langle \phi, \chi_{X_n} \frac{f}{w} w \rangle$

$= \int \chi_{X_n} \frac{f}{w} g = \int \chi_{X_n} f \, u$

For $n \to +\infty \qquad \chi_{X_n} f \xrightarrow{n \to +\infty} f$ in L^1

$\langle \phi, f \rangle = \int f \, u \qquad \forall f \in L^1 \cap L^2$

$|\langle \phi, f \rangle| = \left| \int f \, u \right| \leq |f|_{L^1} \|u\|_{L^\infty}$

$\Rightarrow \quad \|\phi\|_{(L^1)'} \leq \|u\|_{L^\infty}$