Nov 30

Exercise Ω open in $\mathbb{R}^d$ on $u \in L^p(\Omega)$

then $\int_\Omega u \, f \, dx = 0 \quad \forall \; f \in C_c^0(\Omega)$

$\Longrightarrow \quad u = 0$

Corollary $C_c^0(\Omega)$ is dense in $L^p(\Omega)$

for $1 \le p < +\infty$

Pf $Y = \overline{C_c^0(\Omega)}$ is closure in $L^p(\Omega)$

and if $Y \neq L^p(\Omega)$

$\Longrightarrow \exists \; u \in L^{p'}(\Omega)$ s.t.
$u \neq 0$

$$\int f \, u \, dx = 0 \quad \forall \; f \in Y$$

$\Longrightarrow \quad u \equiv 0$ contradiction.

$$(c_0(\mathbb{N}))' = \ell^1(\mathbb{N})$$

<u>Theorem</u> Let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$

$p, q \in [1, \infty]$ $\exists$ $r \geq 1$ s.t.

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

Then consider $f * g(x) := \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy$

This formula defines a function in $L^r(\mathbb{R}^d)$

$$\| f * g \|_{L^r} \leq \| f \|_{L^p} \, \| g \|_{L^q} \quad (1)$$

Young's convolution inequality

<u>Pf</u> The map $(f, g) \in L^p \times L^q \longrightarrow L^r$ (2) will be shown to be a bounded bilinear map and we will show (1)

To bound (1) and the fact that convolution extends to all $L^p \times L^q$ it is enough to show the bound (1) for f and g in appropriate dense subspace of L^p and L^q.

$$\left| \underbrace{\int_{\mathbb{R}^d} dx\, h(x) \int_{\mathbb{R}^d} f(x-y)\, g(y)\, dy}_{I(f,g,h)} \right|$$

$$\leq \quad |h|_{L^{r'}} \; |f|_{L^p} \; |g|_{L^q}$$

$\forall\, h \in L^{r'}$

$$I(f,g,h) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x-y)\, h(x)\, g(y)\, dx\, dy$$

$$f \geq 0 \;, \quad g \geq 0 \;, \quad h \geq 0$$

$$|f|_{L^p} = |g|_{L^q} = |h|_{L^{r'}} = 1$$

$$I(f,g,h) \leq 1$$

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q} \qquad\qquad \frac{1}{r} = 1 - \frac{1}{r'}$$

$$2 = \frac{1}{r'} + \frac{1}{p} + \frac{1}{q}$$

$$\left(2-\frac{1}{p}-\frac{1}{q}\right)r'=1 \qquad \left(2-\frac{1}{p}-\frac{1}{r'}\right)q=1$$

$$\left(2-\frac{1}{r'}-\frac{1}{q}\right)p=1$$

$$\left(1-\frac{1}{p}\right)r'+\left(1-\frac{1}{q}\right)r'=1$$

$$\left(1-\frac{1}{p}\right)q+\left(1-\frac{1}{r'}\right)q=1 \qquad L^{r'}\times L^{q}\times L^{p}\to\mathbb{R}$$

$$\left(1-\frac{1}{r'}\right)p+\left(1-\frac{1}{q}\right)p=1$$

$$I(f,g,h)=\int_{\mathbb{R}^d\times\mathbb{R}^d} f(x-y)\,h(x)\,g(y)\,dx\,dy$$

$$=\int \left(f^p(x-y)\,g^q(y)\right)^{1-\frac{1}{r'}}\left(f^p(x-y)\,h^{r'}(x)\right)^{1-\frac{1}{q}}\left(g^q(y)\,h^{r'}(x)\right)^{1-\frac{1}{p}}dx\,dy$$

$$=\int \underbrace{\left(f^p(x-y)\,g^q(y)\right)^{\frac{1}{r}}}_{L^r}\underbrace{\left(f^p(x-y)\,h^{r'}(x)\right)^{\frac{1}{q'}}}_{L^{q'}}\underbrace{\left(g^q(y)\,h^{r'}(x)\right)^{\frac{1}{p'}}}_{L^{p'}}dx\,dy$$

$$\frac{1}{r}+1=\frac{1}{p}+\frac{1}{q}=1-\frac{1}{p'}+1-\frac{1}{q'}=2-\frac{1}{p'}-\frac{1}{q'}$$

$$\frac{1}{r}+\frac{1}{p'}+\frac{1}{q'}=1$$

Apply Hölder

$$\leq\left(\int dx\,dy\; f^p(x-y)\,g^q(y)\right)^{\frac{1}{r}}\left(\int dx\,dy\; f^p(x-y)\,h^{r'}(x)\right)^{\frac{1}{q'}}$$

$$\left(\int dx\,dy\; g^q(y)\,h^{r'}(x)\right)^{\frac{1}{p'}}=1$$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} dx\, dy\, f^p(x-y)\, g^q(y) =$$

$$= \underbrace{\int_{\mathbb{R}^d} dy\, g^q(y)}_{1} \underbrace{\int_{\mathbb{R}^d} dx\, f^p(x-y)}_{1}$$

$$(f, g) \longrightarrow f * g$$

$$L^p \times L^q \longrightarrow L^r$$

$$f, g \qquad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

$$L^p(\mathbb{R}^d) \qquad g \in L^q(\mathbb{R}^d)$$

$$L^p(\mathbb{R}^d) \ni f \longrightarrow f * g = g * f \in L^r(\mathbb{R}^d)$$

$$\tau_y f(x) = f(x-y)$$

$$\tau_y f(x) = e^{-y \cdot \nabla} f(x) \qquad \tau_y = e^{-y \cdot \nabla}$$

$$\frac{d}{dt} \tau_{ty} f \Big|_{t=0} = \frac{d}{dt} f(x - ty) \Big|_{t=0} = -y \cdot \nabla f(x) \Big|$$

$$t \in \mathbb{R}$$

$$\frac{d}{dt} e^{-t y\cdot\nabla} f \Big|_{t=0} = -y\cdot\nabla f$$

$$\frac{d}{dt} e^{tA} f \Big|_{t=0} = e^{tA} A f \Big|_{t=0} = Af$$

$$\tau_y (g * f) = g * \tau_y f \qquad \forall \quad f.$$

$\mathbb{R}^d \ni h \longrightarrow \tau_h f \in L^p \qquad C^0(\mathbb{R}^d, L^p(\mathbb{R}^d))$

<u>Th</u> (~~Kol~~, R, F)

Let $\mathcal{F} \subseteq L^p(\mathbb{R}^d) \qquad p < +\infty,$

bounded and s.t.

$$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad \text{s.t. } |h| < \delta(\varepsilon) \Rightarrow \|\tau_h f - f\|_{L^p(\mathbb{R}^d)} \leq \varepsilon$$

$$\forall f \in \mathcal{F}.$$

Then for any open bounded Ω in $\mathbb{R}^d$

$\mathcal{F}|_\Omega$ is relatively compact in $L^p(\Omega)$.

<u>Pf</u> We will prove

$\forall \varepsilon > 0 \quad \mathcal{F}|_\Omega$ is contained in a finite union of balls of radius ε in $L^p(\Omega)$

I claim

$$\exists \quad \omega \subset\subset \Omega \quad \text{s.t.} \quad \|f\|_{L^p(\Omega \setminus \omega)} \leq \frac{\varepsilon}{4} \qquad (1)$$
$$\forall \ f \in \mathcal{F}$$

$a, b \in \mathbb{R}_+$

$$T(a,b) = \{ f \in C^1(\mathbb{R}^d) : \|f\|_{L^\infty(\mathbb{R}^d)} \leq a, \ \|\nabla f\|_{L^\infty(\mathbb{R}^d)} \leq b \}$$

$T(a,b)|_\omega$ is relatively compact in $C^0(\omega, \mathbb{R})$

<u>Lemma</u> 1) $\rho \in L^1(\mathbb{R}^d)$ $\quad \int \rho \, dx = 1$

$\rho_\varepsilon(x) = \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$ then

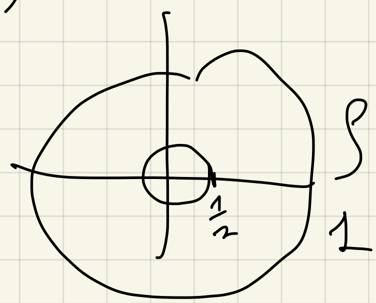
$\forall \ f \in L^p(\mathbb{R}^d) \qquad 1 \leq p < +\infty$

$$\lim_{\varepsilon \to 0^+} \rho_\varepsilon * f = f \quad \text{in} \quad L^p(\mathbb{R}^d)$$

2) $\rho \in C^0_c(\mathbb{R}^d) \implies \rho * f \in C^0(\mathbb{R}^d)$

$\rho, \nabla\rho \in C^0_c(\mathbb{R}^d) \implies \rho * f \in C^1(\mathbb{R}^d)$

$\rho \in C_c^\infty\left(D_{\mathbb{R}^d}(0,1) \, , [0,1] \right)$



$\rho_n(x) = n^d \rho(nx)$

$\rho_{\frac{1}{n}}$ $\qquad \int \rho_n \, dx = 1$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0 \quad$ s.t. $\quad |h| < \delta(\varepsilon) \Rightarrow \| \tau_h f - f \|_{L^p(\mathbb{R}^d)} \leq \varepsilon$ $\quad$ 2

$\forall f \in \mathcal{F}$.

Claim there exists N_ε

s.t. $\quad n > N_\varepsilon \qquad \| \rho_n * f - f \|_{L^p(\mathbb{R}^d)} < \frac{\varepsilon}{4}$

$\forall f \in \mathcal{F}$.

$$\| \rho_n * f - f \|_{L^p(\mathbb{R}^d)} = \left\| \int \rho_n(y) f(x-y) - f(x) \right\|_{L^p(\mathbb{R}^d_x)}$$

$$= \left\| \int dy \, \rho_n(y) \left(f(x-y) - f(x) \right) \right\|_{L^p(\mathbb{R}^d_x)} \leq$$

$$\leq \int_{D_{\mathbb{R}^d}(0,\frac{1}{n})} dy \, \rho_n(y) \, \| \tau_y f - f \|_{L^p(\mathbb{R}^d_x)}$$

~~Supp~~

$$|y| < \frac{1}{n} < \delta\left(\frac{\varepsilon}{4}\right)$$

$$< \frac{\varepsilon}{4} \underbrace{\int dy \, \rho_n(y)}_{1} \qquad \forall f \in \mathcal{F}$$

$$\left|\rho_n * f(x)\right| \leq \int_{\mathbb{R}^d} \rho_n(x-y)\,|f(y)|\,dy$$

$$\leq \underbrace{\|\rho_n(x-\cdot)\|_{L^{p'}(\mathbb{R}^d)}}_{\rho_n} \ \|f\|_{L^p(\mathbb{R}^d)} \leq a_n \quad \forall f \in \mathcal{F}$$

$$\left|\nabla(\rho_n * f)(x)\right| \quad \left|(\nabla\rho_n) * f(x)\right|$$

$$\leq \|\nabla\rho_n\|_{L^{p'}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)} \leq b_n \quad \forall f \in \mathcal{F}$$

$$\{\rho_n * f,\ f \in \mathcal{F}\} \subseteq T(a_n, b_n) \subseteq C^0(\omega) \subseteq L^p(\omega)$$

~~Th~~

$u_1, \dots, u_N \in L^p(\omega)$ st

st $\quad T(a_n, b_n) \subset D_{L^p(\omega)}(u_1, \frac{\varepsilon}{3}) \cup \dots \cup D_{L^p(\omega)}(u_N, \frac{\varepsilon}{3})$

$$\underset{\rho_n * f}{\psi}$$

$$u_j|_{\Omega \setminus \omega} = 0$$

$$\mathcal{F}|_{\Omega} \subset \bigcup_{j=1}^{N} D_{L^p(\Omega)}(u_j, \varepsilon)$$

$$|\rho_n * f - u_1|_{L^p(\Omega)} < \frac{\varepsilon}{3}$$

$$|f - u_1|_{L^p(\Omega)} \leq \underbrace{|f - \cancel{u_1}|_{L^p(\Omega \setminus \omega)}}_{< \frac{\varepsilon}{3}} + \|f - u_1\|_{L^p(\omega)}$$

$$\|f - u_1\|_{L^p(\omega)} = \|f - \rho_n * f + \rho_n * f - u_1\|_{L^p(\omega)}$$

$$\leq \underbrace{\|f - \rho_n * f\|_{L^p(\mathbb{R}^d)}}_{< \frac{\varepsilon}{4}} + \underbrace{\|\rho_n * f - u_1\|_{L^p(\omega)}}_{< \frac{\varepsilon}{3}}$$

claim

$$\exists \ \omega \subset\subset \Omega \quad \text{s.t.} \quad \|f\|_{L^p(\Omega \setminus \omega)} \leq \frac{\varepsilon}{3} \qquad (1)$$

$$\forall \ f \in \mathcal{F}$$

$$\|f\|_{L^p(\Omega \setminus \omega)} \leq \|f - \rho_n * f\|_{L^p(\mathbb{R}^d)} + \|\rho_n * f\|_{L^p(\Omega \setminus \omega)}$$

$$< \frac{\varepsilon}{4} + \left\| \int dy\, \rho_n(x-y) \cdot f(y) \right\|_{L^p(\Omega \setminus \omega)}$$

$$\leq \frac{\varepsilon}{4} + \left\| \int dy\, \rho_n(x-y) |f(y)| \right\|_{L^\infty(\mathbb{R}^d)} |\Omega \setminus \omega|^{\frac{1}{p}}$$

$\leq \frac{\varepsilon}{4} + \|f\|_{L^p(\mathbb{R})} \underbrace{\|\rho_n\|_{L^{p'}(\mathbb{R}^d)}} |\Omega \setminus w|^{\frac{1}{p}}$

i can make this arbitrarily small

I can

$< \frac{\varepsilon}{3}$.