December 2

Prop $f \in C^k_c(\mathbb{R}^d)$ and $g \in L^1_{loc}(\mathbb{R}^d)$

Then $f * g \in C^k(\mathbb{R})$ with

$$\nabla^j (f * g) = \nabla^j f * g \quad \forall \quad j \leq k.$$

Pf $k = 0$ $F(x, y) = f(x-y)\, g(y) \in L^1_x(\mathbb{R}^d_y)$

If $x_n \to x$ in $\mathbb{R}^d$ $\int F(x_n, y)\,dy \to \int F(x, y)\,dy$

$\{x_n\} \subset \overline{D_{\mathbb{R}^d}(x, 1)}$

$K \doteq \operatorname{supp} f$ compact in $\mathbb{R}^d$

$\exists$ K_1 compact in $\mathbb{R}^d$ s.t.

$F(x_n, y)$, $F(x, y)$ have support in y

contained in K_1

$$F(x_n, y) = 1_{K_1}(y)\, F(x_n, y)$$

$$F(x, y) = 1_{K_1}(y)\, F(x, y)$$

$$z = x_n, x \in \overline{D_{\mathbb{R}^d}(\cancel{x}, 1)}$$

$f(z-y)\, g(y) \neq 0$ only if

$z-y \in K$ $\qquad z-y=k \qquad k\in K$

$y = z-k \qquad z \in \overline{D_{\mathbb{R}^d}(x,1)}$

$k \in K$

$\operatorname{supp} f(z-\cdot)\, g(\cdot) \subseteq \underbrace{\overline{D_{\mathbb{R}^d}(0,1)} - K}$ is compact

$D \times K$

$\mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$

$(x_1, x_2) \longrightarrow x_1 - x_2$

$F(x_n, y \qquad x_n \longrightarrow x$

$f(x_n - y)\, g(y) \longrightarrow f(x-y)\, g(y)$

$|f(x_n-y)\, g(y)| \leq |f(x_n-y)\, g(y)|\, 1_{K_1}(y)$

$\leq \|f\|_{L^\infty} \underbrace{|g(y)|\, 1_{K_1}(y)}_{L^1(\mathbb{R}^d_y)}$

By dominated conv

$$\lim_{n\to+\infty} \int f(x_n-y)\, g(y)\, dy = \int \underline{f(x-y) g(y) dy}$$

$\Rightarrow \quad f * g \in C^0(\mathbb{R}^d)$

$k = 1 \qquad f \in C^1_c(\mathbb{R}^d) \qquad g \in L^1_{loc}(\mathbb{R}^d)$

$\nabla (f * g) = \nabla f * g$

$f(x+h-y) - f(\cancel{x}-y) - h \cdot \nabla f(x-y) =$

$= h \cdot I(x-y, h)$

$I(x-y, h) = \int_0^1 [\nabla f(x+sh-y) - \nabla f(x-y)]\, ds$

$\nabla f \in C^0_c(\mathbb{R}^d, \mathbb{R}^d)$ and it is uniformly continuous.

$|I(z, h)| \leq o(1) \qquad o(1)$ dependent only on h with $o(\cancel{h}) \xrightarrow{h \to 0} 0$

Fix on x

$|f(x+h-y) - f(\cancel{x}-y) - h \cdot \nabla f(x-y)| \leq$

$\leq |h| \, |I(x-y, h)| \leq |h| \; o(1) \; \chi_{K_1}(y)$

$|h| \leq 1 \qquad \exists \quad K_1 \quad$ s.t.

$x + h - y \in K$

$x + h - y = k \qquad y = x + h - k$

$$\begin{cases} y \in x + \overline{D_{\mathbb{R}^d}(0,1)} - K \\ y \in x - K \end{cases}$$

$$y \in \left(x + \overline{D_{\mathbb{R}^d}(0,1)} - K\right) \cup (x - K) \subset K_1$$

$$\underline{\left| f*g(x+h) - f*g(x) - h \cdot \nabla f * g(x) \right| =}$$

$$\leq \int \left| \left(f(x+h-y) - f(x-y) - h \cdot \nabla f(x-y) \right) \right| |g(y)| \, dy$$

$$|h| o(1) \underline{\int \chi_{K_1}(y) \, |g(y)| \, dy} \quad \to 0$$

~~for~~ $f * g \in C^1$

$$\nabla (f * g) = \nabla f * g .$$

<u>Th</u> $\rho \in L^1(\mathbb{R}^d)$ $\int \rho(x) \, dx = 1$

$$\rho_\varepsilon(x) \doteq \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$$

$$\rho_\varepsilon * f \xrightarrow{\varepsilon \to 0^+} f \qquad \forall \; f \in L^p(\mathbb{R}^d) \quad p < \infty$$

$$f \in C^0_0(\mathbb{R}^d) \supset \overline{C^0_c(\mathbb{R}^d)} \text{ in } L^\infty(\mathbb{R}^d)$$

$$e^{t\Delta} f \xrightarrow{t \to 0^+} f \qquad \forall f \in L^p(\mathbb{R}^d)$$

$$1 \le p < +\infty$$

Pf $u(t,x) = e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy$

$$\begin{cases} (\partial_t - \Delta)\, u(t,x) = 0 \\ u(0) = f \quad \in L^p(\mathbb{R}^d) \end{cases} \qquad p < +\infty$$

$$\rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}} \qquad \int \rho = 1$$

$$\int e^{-|x|^2} dx = \int e^{-x_1^2} e^{-x_2^2} \cdots e^{-x_d^2} dx =$$

$$= \underbrace{\int e^{-x_1^2} dx_1}_{\sqrt{\pi}} \cdots \int e^{-x_d^2} dx_d$$

$$\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 = \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy =$$

$$= \int_{\mathbb{R}^2} e^{-x^2-y^2} dx\, dy =$$

$$= \int_0^{2\pi} \int_{0}^{+\infty} e^{-r^2} r \, dr \, d\vartheta$$

$$\left(\frac{e^{-r^2}}{-2} \right)'$$

$$= \cancel{2}\pi \quad \frac{1}{\cancel{2}}$$

$$\rho(x) = (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$$

$$(4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} = t^{-\frac{d}{2}} \rho\left(\frac{x}{\sqrt{t}}\right)$$

$$\varepsilon = \sqrt{t} \qquad \rho_{\sqrt{t}}$$

$$\rho_\varepsilon(x) \doteq \varepsilon^{-d} \rho\left(\frac{x}{\varepsilon}\right)$$

$$e^{t\Delta} f = \rho_{\sqrt{t}} * f \xrightarrow{t \to 0^+} f \quad \text{because}$$

of $\rho_\varepsilon * f \xrightarrow{\varepsilon \to 0^+} f$ in $L^p(\mathbb{R}^d)$ (1)

$p < +\infty$

$$\overline{C^0_c(\mathbb{R}^d)} = L^p(\mathbb{R}^d) \qquad f \in C^0_c(\mathbb{R}^d)$$

$$\rho_\varepsilon * f(x) - f(x) = \int \varepsilon^{-d} \rho\left(\frac{x-y}{\varepsilon}\right) f(y) - \underline{\underline{f(x)}}$$

$$= \int \varepsilon^{-d} \rho\left(\frac{y}{\varepsilon}\right) f(x-y) - \int \varepsilon^{-d} \rho\left(\frac{y}{\varepsilon}\right) f(x)\, dy$$

$$= \int \varepsilon^{-d} \rho\left(\frac{y}{\varepsilon}\right) \left[f(x-y) - f(x) \right] dy$$

$y' = \frac{y}{\varepsilon}$, $y = \varepsilon y'$

$$= \int \rho(y) \left[f(x - \varepsilon y) - f(\cancel{x}) \right] dy. \qquad dy = \varepsilon^d dy'$$

$$\Delta(y) = \| f(\cdot - y) - f \|_{L^p(\mathbb{R}^d)}$$

$$\| \rho_\varepsilon * f - f \|_{L^p} = \left\| \int \rho(y) \underbrace{\left[f(\cdot - \varepsilon y) - f \right]} dy \right\|_{L^p}$$

$$\leq \int \rho(y) \, \| f(\cdot - \varepsilon y) - f \|_{L^p}$$

$$= \int \rho(y) \, \Delta(\varepsilon y) \xrightarrow{\varepsilon \to 0^+} 0$$

$$|\Delta(\varepsilon \cancel{y})| \leq 2 \|f\|_{L^p}$$

$$\Delta(\varepsilon y) \xrightarrow{\varepsilon \to 0^+} 0$$

$\forall \; f \in L^p(\mathbb{R}^d)$

$\tau_y f = f(\cdot - y) \longrightarrow f$ in $L^p(\mathbb{R}^d)$

$p < +\infty$

Exercise $\quad f, g \in C^0_c(\mathbb{R}^d)$

$$\Rightarrow \quad \operatorname{supp} f * g \subseteq \overline{\operatorname{supp} f + \operatorname{supp} g}$$

Prop $\forall$ Ω open set in $\mathbb{R}^d$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, for $1 \le p < +\infty$

Pf $\Omega = \mathbb{R}^d$

$$\rho \in C_c^\infty(\mathbb{R}^d, [0,1]) \qquad \int \rho = 1$$

$$\operatorname{supp} \rho \subseteq D_{\mathbb{R}^d}(0,1) \qquad \rho_\varepsilon \qquad \varepsilon > 0$$

I know that $C_c^0(\Omega)$ is dense in $L^p(\Omega)$. $p < +\infty$

$$g \in C_c^0(\mathbb{R}^d) \qquad K = \operatorname{supp} g$$

$$\rho_\varepsilon * g \xrightarrow{\varepsilon \to 0^+} g \quad \text{in} \quad L^p(\mathbb{R}^d)$$

$$\operatorname{supp}(\rho_\varepsilon * g) \subseteq \overline{\operatorname{supp} \rho_\varepsilon + K} =$$

$$\subseteq \overline{D_{\mathbb{R}^d}(0,\varepsilon) + K} \quad \text{is compact}$$

$$= \{ y \in \mathbb{R}^d : \operatorname{dist}(y, K) \le \varepsilon \} = K_\varepsilon$$

is compact

$$\rho_\varepsilon * g \in C_c^\infty(\mathbb{R}^d) \qquad \operatorname{supp} \rho_\varepsilon * g \subseteq K_\varepsilon$$

$\Omega \subsetneq \mathbb{R}^d$

$g \in C^0_c(\Omega)$ $\qquad K = \operatorname{supp} g$

$\cancel{K}$ $\qquad \gamma := \operatorname{dist}(K, \partial\Omega) > 0$

For $0 < \varepsilon < \gamma \Rightarrow$ $\boxed{K_\varepsilon \subset\subset \Omega}$

$$\underbrace{\rho_\varepsilon * g}_{C^\infty_c(\mathbb{R}^d)} \xrightarrow[L^p(\mathbb{R}^d)]{\varepsilon \to 0^+} g$$

$C^\infty_c(\Omega)$

Hilbert

Def H a v. space on $\mathbb{R}$ is called pre-Hilbert if it has a bilinear symmetric form $(u, v)_H$ $u, v \in H$, strictly positive, that is

$(u, u)_H \geq 0 \quad \forall u$ and

$(u, u)_H = 0 \iff u = 0$

$\|u\|_H = \sqrt{(u, u)_H}$,

If it is complete for $\|\cdot\|_H$, then it is a Hilbert space

Def H vector space on $\mathbb{C}$, $B(u,v) = (u,v)_H$

$$B : H \times H \longrightarrow \mathbb{C} \xrightarrow{Re} \mathbb{R}$$

B is sesquilinear

$$B(\lambda x + \mu y, z) = \lambda B(x,z) + \mu B(y,z)$$

$$B(z, \lambda x + \mu y) = \bar{\lambda} B(z,x) + \bar{\mu} B(z,y)$$

B Hermitian

$$B(x,y) = \overline{B(y,x)}$$

$Re \circ B$

B positive

$$B(x,x) \geq 0 \quad \forall x \in H$$

Nondegenerate if

$$B(x,x) = 0 \implies x = 0$$

$$(u,v)_H = B(u,v)$$

$$\|u\|_H = (u,u)_H^{\frac{1}{2}}$$

$f, g \in L^2(X, d\mu)$

$$(f,g)_{L^2} = \int_X f(x)\,\overline{g(x)}\,d\mu(x)$$

$$L^2(\mathbb{R}^d, \mathbb{C})$$

$$\mathrm{Re} \int_X f(x)\,\overline{g(x)}\,dx$$

$$\left\| \frac{a+b}{2} \right\|_H^2 + \left\| \frac{a-b}{2} \right\|_H^2 = \frac{1}{2} \left(\|a\|_H^2 + \|b\|_H^2 \right)$$

$$\|a+b\|_H^2 + \|a-b\|_H^2 = 2\left(\|a\|_H^2 + \|b\|_H^2 \right)$$

$$|(a,b)_H| \leq \|a\|_H \, \|b\|_H$$

$$2(a,b)_H + 2(b,a)_H = \|a+b\|_H^2 - \|a-b\|_H^2$$

$$= (a+b, a+b) - (a-b, a-b)$$

$$= (a,a) + (b,b) + (a,b) + (b,a) - (a,a) - (b,b)$$

$$+ (a,b) + (b,a) =$$

$$2(a,b)_H + 2(b,a)_H = \|a+b\|_H^2 - \|a-b\|_H^2 \le \|a+b\|_H^2 + \|a-b\|_H^2$$

$$= 2\|a\|_H^2 + 2\|b\|_H^2$$

$$(a,b)_H + (b,a)_H \le \|a\|_H^2 + \|b\|_H^2$$

$$\left|2\,\mathrm{Re}(a,b)_H\right| \le \lambda^2 \|a\|_H^2 + \|b\|_H^2 \frac{1}{\lambda^2} = 2\|a\|_H \|b\|_H$$

$$\lambda^2 = \frac{\|b\|_H}{\|a\|_H}$$

$$\cancel{2}\left|\mathrm{Re}(a,b)_H\right| \le \cancel{2}\|a\|_H \|b\|_H$$

$$|(a,b)_H| \le \|a\|_H \|b\|_H$$

$$(a,b)_H = e^{i\vartheta_0} r$$

$$(e^{-i\vartheta_0} a, b)_H = r$$

$$|(a,b)_H| = r = (e^{-i\vartheta_0} a, b)_H$$

$$\le \|\cancel{e^{-i\vartheta_0}} a\|_H \|b\|_H$$