2 dicembre $(1+x)^a$

$$(1+x)^a = P_n(x) + o(x^n)$$

$$c\,o(x^n) = o(x^n)$$

$$= o(c\,x^n)$$

$$\lim_{x\to 0} \frac{c\,o(x^n)}{x^n} = 0$$

$$\lim_{x\to 0} \frac{\cancel{c}\,o(x^n)}{\cancel{c}\,x^n} = 0$$

$$(1+x)^a = \sum_{j=0}^{n} \binom{a}{j} x^j + o(x^n)$$

$$(1-x)^a = \sum_{j=0}^{n} \binom{a}{j} (-1)^j x^j + \underbrace{o((-x)^n)}_{o(x^n)}$$

$$\frac{1}{1-x} = (1-x)^{-1}$$

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x}$$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + \underbrace{\frac{x^{n+1}}{1-x}}_{o(x^n)}$$

perché $\displaystyle \lim_{x\to 0} \frac{\frac{x^{n+1}}{1-x}}{x^n} = \lim_{x\to 0} \frac{x^{\cancel{n}+1}}{(1-x)\,\cancel{x^n}}$

$$= \lim_{x\to 0} \frac{x}{1-x} = \frac{0}{1} = 0$$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + o(x^n) \implies P_n(x) = \sum_{j=0}^{n} x^j$$

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-1)^j x^j + o(x^n)$$

$$\frac{1}{1+x^2} = \sum_{j=0}^{n} (-1)^j x^{2j} + o(x^{2n})$$

$$P_{2n}(x) = P_{2n+1}(x)$$

Esercizio $\quad f(x) = \dfrac{x^3}{1-x^3}$ calcola $f^{(\ell)}(0) \quad \forall\, \ell$

$$\frac{1}{1-y} = \sum_{j=0}^{n} y^j + o(y^n)$$

$$\frac{1}{1-x^3} = \sum_{j=0}^{n} x^{3j} + o(x^{3n}) \qquad x^3 o(x^{3n}) = o(x^{3n+3})$$

$$\frac{x^3}{1-x^3} = \underbrace{\sum_{j=0}^{n} x^{3j+3}}_{P_{3n+3}} + o(x^{3n+3})$$

$$P_{3n+3}(x) = \sum_{k=0}^{3n+3} \frac{f^{(k)}(0)}{k!} x^k$$

se k non è della forma ~~k=~~ $3j+3$ per $j=0,..,n$

$f^{(k)}(0) = 0$

se $k = 3j+3$ per ~~j=~~ $j = 0,\dots,n$

$$\frac{f^{(3j+3)}(0)}{(3j+3)!} = 1 \qquad f^{(3j+3)}(0) = (3j+3)! \quad \forall\, j = 0,1,\dots$$

$$\lim_{x\to 0^+} \frac{\sqrt{1+2x^a} - 1 - x^2}{\lg(1+x+x^2) - x} \qquad \frac{0}{0} \qquad a \in \mathbb{R}_+$$

$$\frac{A_1\, x^{A_2}}{B_1\, x^{B_2}}$$

denomin

$$\lg(1+y) = y - \frac{y^2}{2} + o(y^2)$$

$$\lg(1+x+x^2) = x+x^2 - \frac{(x+x^2)^2}{2} + o\left((x+x^2)^2\right)$$

$$o\left((x+x^2)^2\right) = o(x^2)$$

$$o\left((x+x^2)^2\right) = o\left(x^2(1+\cancel{x})^2\right) = o\left(x^2(1+o(1))\right) = o(x^2)$$

$$\boxed{\lim_{x\to 0} \frac{o\left(x^2(1+o(1))\right)}{x^2} = 0}$$

$$0 = \lim_{x\to 0} \frac{o\left(x^2(1+o(1))\right)}{x^2\,(1+o(1))} = \lim_{x\to 0} \frac{o\left(x^2(1+o(1))\right)}{x^2} \quad \cancel{\frac{1}{1+o(1)}}$$

$$\underset{\swarrow\, 0}{\frac{o\left(x^2(1+o(1))\right)}{x^2}} = \frac{\overset{\nearrow 0}{o\left(x^2(1+o(1))\right)}}{x^2(1+o(1))} \quad \overset{\nearrow 1}{(1+o(1))}$$

$$\lg(1+x+x^2) = x+x^2 - \frac{(x+x^2)^2}{2} + o(x^2)$$

$$= x+x^2 - \frac{x^2+2x^3+x^4}{2} + o(x^2)$$

$$= x + \frac{1}{2}x^2 + o(x^2)$$

$$\text{denomin} = \lg(1+x+x^2) - x = \cancel{x} + \frac{1}{2}x^2 - \cancel{x} + o(x^2)$$

$$= \frac{1}{2}x^2 + o(x^2)$$

$$= o(x)$$

$$\lim_{x\to 0^+} \frac{\sqrt{1+2x^a} - 1 - x^2}{\frac{1}{2}x^2 + \cancel{o(x^2)} \quad \cancel{a_3x^3 + a_4x^4 + \dots + a_{100}x^{100} + o(x^{100})}}$$

$$\frac{1}{2}x^2 + a_3x^3 + \dots + a_{100}x^{100} + o(x^{100}) =$$

$$= \frac{1}{2}x^2 \left(1 + \underbrace{\frac{a_3x^3}{\frac{1}{2}x^2} + \dots + \frac{a_{100}x^{100}}{\frac{1}{2}x^2} + \frac{o(x^{100})}{\frac{1}{2}x^2}}_{o(1)}\right)$$

$$= \frac{1}{2}x^2 \left(\cancel{1 + o(1)}\right)$$

$$\text{num} = \sqrt{1+2x^a} - 1 - x^2$$

$$(1+y)^{\frac{1}{2}} = 1 + \frac{1}{2}y + \binom{\frac{1}{2}}{2} y^2 + o(y^2) =$$

$$= 1 + \frac{1}{2}y + \frac{\frac{1}{2}\overbrace{\left(\frac{1}{2}-1\right)}^{-\frac{1}{2}}}{2} y^2 + o(y^2)$$

$$= 1 + \frac{1}{2}y - \frac{1}{8}y^2 + o(y^2)$$

$$(1+2x^a)^{\frac{1}{2}} = 1 + x^a - \frac{1}{2}x^{2a} + o(x^{2a})$$

$$\text{num} = (1+2x^a)^{\frac{1}{2}} - 1 - x^2 = x^a - \frac{1}{2}x^{2a} + o(x^{2a}) - x^2$$

$$\text{num} = x^a - \frac{1}{2}x^{2a} - x^2 + o(x^{2a}) = x\left(x^{a-1} - \frac{1}{2}x^{2a-1} - x + o(x^{2a-1})\right) \quad a > 0$$

$$\text{num} = x^a + o(x^a) = x^a(1+o(1)) \qquad 0 < a < 2$$

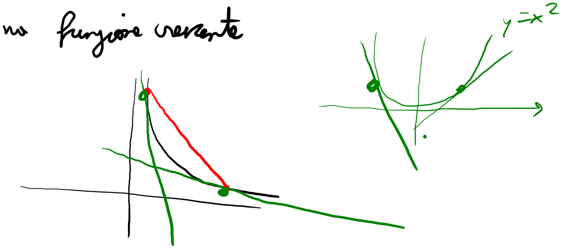
$$\text{num} = -x^2 + o(x^2) \qquad a > 2$$

per $a = 2$

$$\text{num} = -\frac{1}{2}x^4 + o(x^4)$$

Teor Sia $f:(a,b)\to\mathbb{R}$ t.c. $f'(x)$ esiste $\forall x\in(a,b)$.
Sono equivalenti:

1) f è convessa
2) $f'(x)$ è una funzione crescente



Es $f(x)=x^2$ $\qquad f'(x)=2x$

Teor Sia $f:(a,b)\to\mathbb{R}$ e supponiamo che $f''(x)$ sia definita $\forall x\in(a,b)$. Sono equivalenti

1) f è convessa in (a,b)
2) $f''(x)\geq 0 \quad \forall x\in(a,b)$

$$\cancel{f''} \quad (x^2)''=((x^2)')'=(2x)'=2>0$$

$$f(x)=e^{x^2} \qquad f'(x)=e^{x^2}\,2x$$

$$f''=\left(e^{x^2}2x\right)'=\boxed{(e^{x^2})'}2x+e^{x^2}(2x)'=$$

$$=e^{x^2}(2x)^2+e^{x^2}2=2e^{x^2}(x^2+1)>0$$

$$\lim_{x \to 0^+} \frac{num}{den} = \lim_{x \to 0^+} \begin{cases} \dfrac{x^a}{\frac{1}{2} x^2} = +\infty & 0 < a < 2 \\[2ex] \dfrac{-x^2}{\frac{1}{2} x^2} = -2 & a > 2 \\[2ex] \dfrac{-\frac{1}{2} x^4}{\frac{1}{2} x^2} = 0 & \end{cases}$$

$$\frac{num}{den} = \begin{cases} 2 \quad x^{a-2} \left(1 + o(1)\right) & a < 2 \\[2ex] -2 \quad \left(1 + o(1)\right) & a = 2 \end{cases}$$