AG 3 - optional exercises

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1. Consider the rational map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad (a_0 : a_1 : a_2) \to (a_1 a_2 : a_0 a_2 : a_0 a_1).$$

   (a) Determine the domain of f.

   (b) Show that f cannot be extended to $\mathbb{P}^2$.

   (c) Show that f is birational and it is its own inverse. This is called the **Cremona transformation**.

2. Let $Y = V_P(f(x, y, z)) \subset \mathbb{P}^2$ be a smooth plane projective curve of degree $d > 1$. Let $C(Y) = V(fx, y, z)) \subset \mathbb{A}^3$ be the affine cone of Y. Consider the blow-up $\mathrm{Bl}_{(0,0,0)}\mathbb{A}^3$, and let $E \cong \mathbb{P}^2$ be the exceptional divisor. Show that the strict transform $\widetilde{C(Y)}$ of $C(Y)$ satisfies the property that $\widetilde{C(Y)} \cap E$ is isomorphic to Y.

3. $\star$ (very difficult) Consider the rational map

$$f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^3, \quad (s : t : u) \to (s^2 t - st^2 : s^2 u - su^2 : t^2 u - tu^2 : stu).$$

   (a) determine the domain $D(f)$ of f;

   (b) find the equation of the projective surface $S = \overline{f(\mathbb{P}^2)} \subset \mathbb{P}^3$; is it familiar?

   (c) we have, by definition, that $\overline{\Gamma_f} = \mathrm{Bl}_Z$, where $Z = \mathbb{P}^2 \setminus D(f)$; find the equations in $\mathbb{P}^3$ of the images of the exceptional divisors under the second projection $p_2 : \mathbb{P}^2 \times \mathbb{P}^3 \to \mathbb{P}^3$.

4. Compute the dimension of the locus of reducible hypersurfaces of degree $d \geq 2$ in $\mathbb{P}^n$ for $n \geq 2$.

5. Singular plane curves: let $\mathbb{P}^N = \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_d)$ with $d \geq 2$, and set

$$X := \{(p, [F]) \mid p \in V(F)_{sing} \subset \mathbb{P}^2\} \subset \mathbb{P}^2 \times \mathbb{P}^N.$$

   - Prove that X is closed in $\mathbb{P}^2 \times \mathbb{P}^N$;
   - consider the first projection $p_1 : X \to \mathbb{P}^2$ and prove that $\dim X = N - 1$;
   - consider the fibers of the second projection $p_2 : X \to \mathbb{P}^N$ and prove that the locus in $\mathbb{P}^N$ corresponding to singular curves is a hypersurface.