FIRST ASSIGNMENT; EX: 6, 7, 8

6) $f : X \longrightarrow Y$ continuous map, then:

$$X \text{ IRREDUCIBLE} \Longrightarrow f(X) \text{ IRREDUCIBLE}$$

<u>proof</u>: $f(X)$ REDUCIBLE $\Longrightarrow \exists\, \emptyset \neq C_1, C_2 \subsetneq f(X)$:

$f(X) = C_1 \cup C_2 \Longrightarrow X = f^{-1}(f(X)) = f^{-1}(C_1) \cup f^{-1}(C_2)$

f continuous $\Longrightarrow f^{-1}(C_i)$ closed in X $\forall i = 1, 2$

Furthermore $f^{-1}(C_i) \neq \emptyset$ and $f^{-1}(C_i) \subsetneq X$ $\forall i = 1, 2$ because f is clearly surjective on it's image. $\Longrightarrow X$ REDUCIBLE $\blacksquare$

FIRST ASSIGNMENT ; EX : 6, 7, 8

7) $X \subset Y \subset \mathbb{A}^n$ Zar.-closed subsets then:
"every irreducible components of X is contained in an irreducible components of Y."

<u>proof</u>: $\mathbb{A}^n$ Noeth. $\Longrightarrow X, Y$ Noeth (because they are closed)

$\Longrightarrow X = X_1 \cup \ldots \cup X_r$ and $Y = Y_1 \cup \ldots \cup Y_s$.

$\Longrightarrow \forall i \in \{1, \ldots, r\}$ we have $X_i = X_i \cap Y = (Y_1 \cap X_i) \cup \ldots \cup (Y_s \cap X_i)$

But X_i's are IRREDUCIBLE COMPONENTS

$$\Longrightarrow \quad Y_j \cap X_i = \begin{cases} X_i \\ \text{or} \\ \emptyset \end{cases} \quad \text{and since } X_i \neq \emptyset$$

$\exists j \in \{1, \ldots, s\} : \; Y_j \cap X_i = X_i \Longrightarrow \exists j \in \{1, \ldots, s\} : X_i \subset Y_j$ ∎

FIRST ASSIGNMENT ; EX : 6, 7, 8

8) An example of two irreducible sets whose intersection is reducible.

$V(Y), V(Y-XZ)$ are irreducible because

$$\mathbb{K}[X,Y,Z]/I(V(Y)) = \mathbb{K}[X,Y,Z]/(Y) \cong \mathbb{K}[X,Z] \text{ is a domain}$$

$\Longrightarrow I(V(Y))$ is prime $\Longrightarrow V(Y)$ is irred.

And $I(V(Y-XZ)) = \sqrt{Y-XZ} = (Y-XZ)$ because $Y-XZ$ is irreducible and since $\mathbb{K}[X,Y,Z]$ is an UFD we have also irreducible = prime $\Longrightarrow (Y-XZ)$ is prime $\Longrightarrow V(Y-XZ)$ is irred.

But $V(Y) \cap V(Y-XZ) = \underbrace{(V(Y) \cap V(X)) \cup (V(Y) \cap V(Z))}_{\text{closed and proper subsets}}$

$\Longrightarrow$ they're intersection is REDUCIBLE.

FIRST ASSIGNMENT ; EX 1 (Another proof)

1) let $\phi : \mathbb{A}^1 \to \mathbb{A}^3$, $t \mapsto (t, t^2, t^3)$; prove ϕ to be an homeomorphism on its image.

<u>proof</u>:

- ϕ is clearly injective and surjective on its image.
  $\Longrightarrow \phi$ is bijective.
- ϕ is closed because the closed subsets of $\mathbb{A}^1$ are just $\emptyset, \mathbb{A}^1$ and finites unions of points and $\phi(\emptyset) = \emptyset$, $\phi(\mathbb{A}^1) = \mathrm{Im}(\phi) =$
  $= V(X^2 - Y, X^3 - Z)$, $\phi(\bigcup_{i=1}^{n} \{t_i\}) = \bigcup_{i=1}^{n} \{\phi(t_i)\} = \bigcup_{i=1}^{n} \{(t_i, t_i^2, t_i^3)\}$ are Zar closed in $\mathbb{A}^3$ (the points in $\mathbb{A}^3$ are intersections of 3 planes).
- continuity: let C Zar-closed in $\mathbb{A}^3$ i'll show $\phi^{-1}(\mathrm{Im}(\phi) \cap C)$ to be Zar-closed in $\mathbb{A}^1$.
  $C = V(f_0, \dots, f_k)$ for some $f_0, \dots, f_k \in \mathbb{K}[X, Y, Z]$.
  $\Longrightarrow \mathrm{Im}(\phi) \cap C = V(X^2 - Y, X^3 - Z, f_0, \dots, f_k) = V(X^2 - Y, X^3 - Z, f_0) \cap$
  $V(f_1, \dots f_k) \underset{(*)}{=} V(X^2 - Y, X^3 - Z, f_1, \dots, f_k) = \dots \text{iterating} \dots = V(X^2 - Y, X^3 - Z)$

  $(*)$: $V(f_i, X^2 - Y, X^3 - Z) = V(X^2 - Y, X^3 - Z)$ $\forall i \in \{0, \dots, k\}$
  if $V(f_i) \cap \mathrm{Im}(\phi) \neq \emptyset$ and $V(f_i) \cap \mathrm{Im}(\phi) \neq \bigcup_{i=1}^{n} \{p_i\}$, $p_i \in \mathbb{A}^3$

$\Longrightarrow \mathrm{Im}(\phi) \cap C$ can only be $\begin{cases} \emptyset \\ \text{finite union of points} \\ \mathrm{Im}(\phi). \end{cases}$

so that his preimage is closed $\Longrightarrow \phi$ is continuous. ∎

<u>proof of (*)</u>: let $f_i \in \mathbb{K}[X, Y, Z]$ and consider the following rings homomorphism
$$\varphi : \mathbb{K}[X, Y, Z] \longrightarrow \mathbb{K}[S]$$
$$f(X, Y, Z) \longmapsto f(S, S^2, S^3).$$

$\Longrightarrow \ker(\varphi) = \{f \in \mathbb{K}[X, Y, Z] : f(P) = 0 \ \forall P \in \mathrm{Im}(\phi)\} = I(\mathrm{Im}(\phi))$.

Now $\varphi(f_i) = f_i(S, S^2, S^3)$ and for the hypothesis of $(*)$ it has infinite roots $\Longrightarrow f_i(S, S^2, S^3) \equiv 0 \Longrightarrow f_i \in \ker(\varphi)$

$\Longrightarrow (f_i) \subset I(\mathrm{Im}(\phi)) \Longrightarrow V(f_i) \supset \mathrm{Im}(\phi)$.

$\Longrightarrow \mathrm{Im}(\phi) \subset V(f_i) \cap \mathrm{Im}(\phi) = V(f_i, X^2 - Y, X^3 - Z) \subset \mathrm{Im}(\phi)$.

$\Longrightarrow V(f_i, X^2 - Y, X^3 - Z) = V(X^2 - Y, X^3 - Z)$ ∎

<u>Remark</u>: of course if $\exists j : V(f_j, X^2 - Y, X^3 - Z) = \begin{cases} \emptyset \\ \text{finite union of points} \end{cases}$ then:

$V(f_0, \dots, f_j, \dots f_k) \cap \mathrm{Im}(\phi) = \begin{cases} \emptyset \\ \text{finite union of points} \end{cases}$.