① Let $\Phi : \mathbb{A}^1_k \to \mathbb{A}^3_k$ defined as $\Phi(t) = (t, t^2, t^3)$. Show that Φ is an homeomorphism with its image

Solution: Φ is clearly a bijection: it is enough to look at the first component for injectivity and it is clearly surjective if we restrict it to its image.

- We show that Φ is closed: let $\alpha \subseteq K[x]$ be an ideal, $V(\alpha) \subseteq \mathbb{A}^1$ Zariski closed. Since $k[x]$ is a PID, $\exists f \in k[x]$ s.t. $\alpha = \langle f \rangle$.
Then $V(\alpha) = V(f)$ and $\Phi(V(f)) = \{ \Phi(a) : f(a) = 0 \} = \Phi(\mathbb{A}^1) \cap V(\hat{f})$ where $\hat{f}(x, y, z) = f(x)$.

- We finally prove Φ to be continuous: to prove this we first observe that $\Phi(\mathbb{A}^1) = V(x^2 - y, x^3 - z)$. Let $V(\beta) \subseteq \mathbb{A}^3$ be Zariski closed, $\beta \subseteq k[x_1, x_2, x_3]$ an ideal, then:

$$\cancel{V(\beta) \cap V(x^2 - y, x^3 - z) = V(\{ f(x, x^2, x^3) : f \in \beta \}) \cap V(x^2 - y, x^3 - z)}$$

$$\cancel{\text{and: } \Phi^{-1}(V(\beta)) = \Phi^{-1}(V(\beta)) \cap \underbrace{\Phi^{-1}(\Phi(\mathbb{A}^1))}_{= \mathbb{A}^1} = \Phi^{-1}(V(\beta) \cap \Phi(\mathbb{A}^1)) =}$$

$$\cancel{= \Phi^{-1}(V(\{ f(x, x^2, x^3) : f \in \beta \})) = \{ a \in \mathbb{A}^1 : f(\Phi(a)) = 0 \ \forall f \in \beta \} =}$$

$$\cancel{= V(f(\Phi(x)))} \quad V(\{ f(x, x$$

and $\Phi^{-1}(V(\beta)) = \{ a \in \mathbb{A}^1 : f(a, a^2, a^3) = 0 \ \forall f \in \beta \} =$

$= V(\{ f(t, t^2, t^3) : f(x, y, z) \in \beta \})$ is Zariski closed.

② $X = V(\alpha) = \{x = (x_1, \dots, x_m) \in \mathbb{A}^m \mid F(x) = 0 \ \forall F \in \alpha\}$

WITH $\alpha \subseteq \mathbb{K}[x_1, \dots, x_m]$ IDEAL.

$p \notin X \Rightarrow \exists\, g \in \alpha$ SUCH THAT $g(p) = c \in \mathbb{K} \setminus \{0\}$ (OTHERWISE $p \in X$).

$\mathbb{K}$ IS A FIELD $\Rightarrow \exists\, c^{-1} \in \mathbb{K}$ SUCH THAT $c^{-1}c = 1$.

SET $f := c^{-1} g \in \mathbb{K}[x_1, \dots, x_m]$, WE HAVE $f \in \alpha$ BECAUSE α IS AN IDEAL.

$f \in \alpha$, SO $f(x) = 0 \ \forall x \in X \Rightarrow f_{|X} \equiv 0$.

BUT $f(p) = c^{-1} g(p) = c^{-1} c = 1$. $\square$

③ $f : \mathbb{A}^m \longrightarrow \mathbb{A}^1$, $f \in \mathbb{K}[x_1, \dots, x_m]$, IS CONTINUOUS IF AND ONLY IF $\forall C \subseteq \mathbb{A}^1$ ZARISKI CLOSED SUBSET $f^{-1}(C)$ IS ZARISKI CLOSED IN $\mathbb{A}^m$. WE KNOW THAT ZARISKI CLOSED IN $\mathbb{A}^1$ ARE $\emptyset$, $\mathbb{A}^1$ AND FINITE SET OF POINTS, $f^{-1}(\emptyset)$ AND $f^{-1}(\mathbb{A}^1)$ ARE TRIVIALLY CLOSED IN $\mathbb{A}^m$. THE PREIMAGE OF A FINITE SET OF POINTS IS THE UNION OF THE ~~PREIMAGES~~ PREIMAGES OF EVERY SINGLE POINT. SINCE THE FINITE UNION OF CLOSED IS CLOSED, IT'S ENOUGH TO SHOW THAT $f^{-1}(p) \subseteq \mathbb{A}^m$ IS ZARISKI CLOSED $\forall p \in \mathbb{A}^1$. WE CAN IDENTIFY $\mathbb{A}^1$ WITH THE FIELD $\mathbb{K}$, SO WE HAVE TO SHOW THAT $f^{-1}(k)$ IS ZARISKI CLOSED IN $\mathbb{A}^m$ $\forall k \in \mathbb{K}$. IF WE SET $g := f - k \in \mathbb{K}[x_1, \dots, x_m]$ WE HAVE

$$f^{-1}(k) = \{x = (x_1, \dots, x_m) \in \mathbb{A}^m \mid f(x) = k\} = \{x = (x_1, \dots, x_m) \in \mathbb{A}^m \mid g(x) = 0\} = V(g)$$

WHICH IS ZARISKI CLOSED IN $\mathbb{A}^m$. $\square$