① Let $\phi: \mathbb{A}^2_k \to \mathbb{A}^2_k$ $(k = \bar{k})$. Find the image of ϕ
$(x, y) \to (x, xy)$

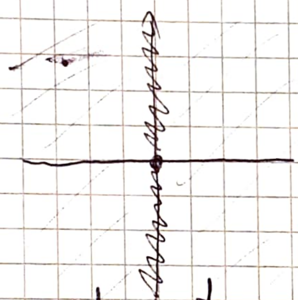
and determine whether it is open, closed or dense.

Solution: We first determine $\phi(\mathbb{A}^2)$:

$$\phi(\mathbb{A}^2) = \phi(\mathbb{A}^2 \setminus V(x)) \cup \phi(\underset{V(x)}{\cancel{\mathbb{A}^2 \setminus V(y)}}) = \{(a,b) : a \neq 0\} \cup \{(0,0)\}:$$

indeed, if $x \neq 0$ and we fix $(a,b) \in \mathbb{A}^2 \setminus V(x)$, with $y = a^{-1}b$ we get $\phi(x,y) = (a,b)$, then $\phi(\mathbb{A}^2 \setminus V(x)) \supseteq \{(a,b) : a \neq 0\}$; the other inclusion is obvious with the null factor law.

Graphically: $\phi(\mathbb{A}^2)$:



- $\phi(\mathbb{A}^2)$ is not open: if it were, $\phi(\mathbb{A}^2) \cap V(x) = \{0,0\}$ would be open in $V(x) \simeq \mathbb{A}^1$ and we get a contradiction.
- $\phi(\mathbb{A}^2)$ is not closed: if it were, $\phi(\mathbb{A}^2) \cup V(x)$ would be a composition of $\mathbb{A}^2$ into proper closed non empty set
- $\phi(\mathbb{A}^2)$ is dense, since $\mathbb{A}^2 \setminus V(x) \subseteq \phi(\mathbb{A}^2)$ is an open in an irreducible and is therefore dense.

② Let $\phi: \mathbb{A}^3 \to \mathbb{A}^3$. Same requests as the previous exercise
$(x, y, z) \to (x, xy, xyz)$

Solution: the same exact reasoning still applies