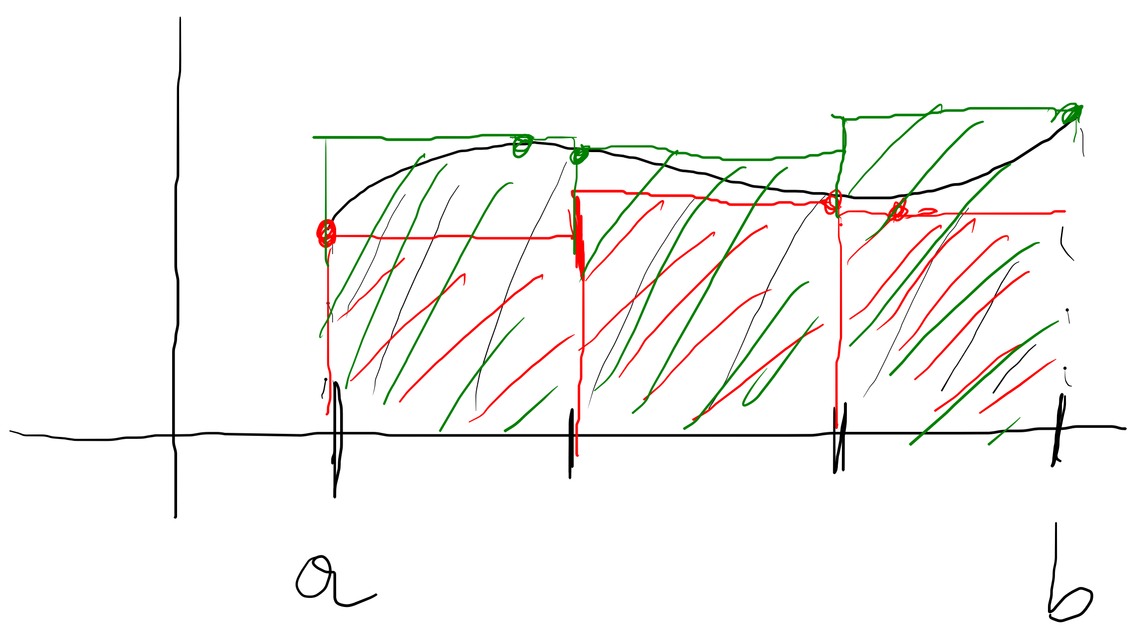
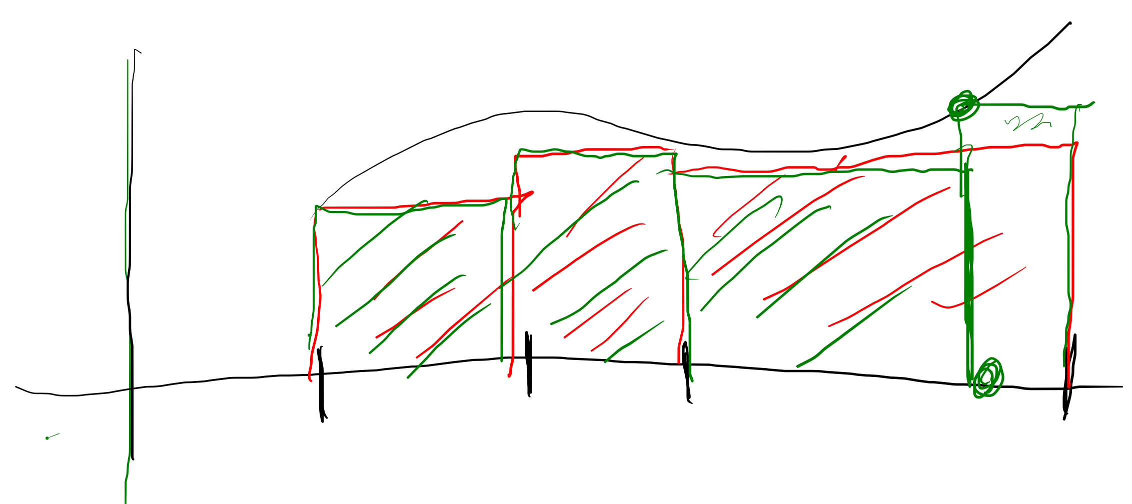
6 Dicembre pomeriggio.

Integrale

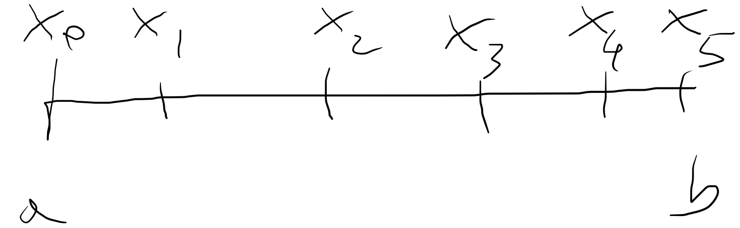


$$y = f(x)$$

$$\int_a^b f(x)\,dx$$



$[a,b]$



Una scomposizione

Δ

$$x_0 = a < x_1 < \dots < x_n = b$$

$$[x_0, x_1],\ [x_1, x_2],\ [x_2, x_3],\ \dots\ [x_{n-1}, x_n]$$

$$\left|[x_{j-1}, x_j]\right| = x_j - x_{j-1}$$

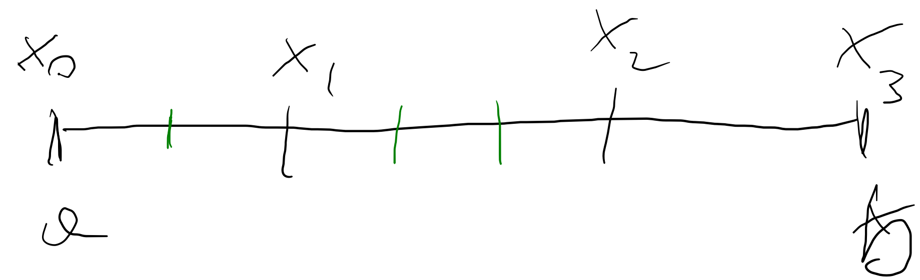
Calibro di Δ è

$$|\Delta| = \min\{x_j - x_{j-1} \,;\, j = 1, \dots, n\}$$

$$\sum_{j=1}^{n} (x_j - x_{j-1}) = x_n - x_0 = b - a$$

$$(\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + \dots\dots + (\cancel{x_{n-1}} - \cancel{x_{n-2}}) + (x_n - \cancel{x_{n-1}})$$

Raffinamento



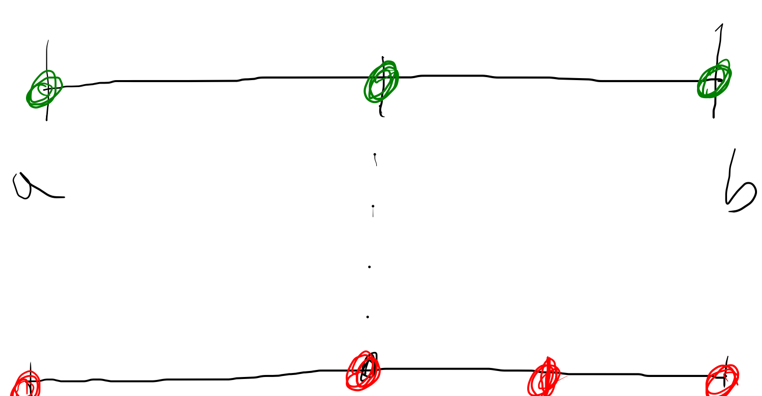
Se Δ, $x_0 = a < x_1 < \dots < x_n = b$, Δ' sarà una decomposizione più raffinata di Δ se

$$\Delta' \qquad y_0 = a < y_1 < \dots < y_N = b$$

se $N \geq n$ e se ogni x_j si può ritrovare tra i punti y

$$\{x_0, \dots, x_n\} \subseteq \{y_0, \dots, y_N\}$$

Scriverò $\Delta' \preceq \Delta$



Fissiamo $f:[a,b]\longrightarrow \mathbb{R}$.

Chiediamo che sia limitata, cioè

$$f([a,b]) = \{ f(x) \ ; \ a \le x \le b \}$$

è limitato, cioè $\exists$ $m < M$ in $\mathbb{R}$ t.c.

$$m \le f(x) \le M \quad \text{per ogni } x \in [a,b]$$

Consideriamo una decomposizione di $[a,b]$

Δ $\quad x_0 = a < x_1 < \dots < x_n = b$

$$f([x_{j-1},x_j]) \subseteq f([a,b])$$

$S(\Delta) =$

$$\inf f([x_{j-1},x_j]) \ge \inf f([a,b])$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf f([x_{j-1},x_j])$$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup f([x_{j-1},x_j])$$

$$\sup f([x_{j-1},x_j]) \le \sup f([a,b])$$

Lemma Se $m \le f(x) \le M$ $\forall x \in [a,b]$, si ha

$$m(b-a) \le s(\Delta) \le S(\Delta) \le M(b-a)$$

Dim

$$\inf f([x_{j-1},x_j]) \le \sup f([x_{j-1},x_j]) \quad \forall j$$

$$(x_j - x_{j-1}) \inf f([x_{j-1},x_j]) \le (x_j - x_{j-1}) \sup f([x_{j-1},x_j])$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf f([x_{j-1},x_j]) \le \sum_{j=1}^{n} (x_j - x_{j-1}) \sup f([x_{j-1},x_j]) = S(\Delta)$$

$$f(x) \le M \quad \forall x \in [a,b]$$

$$f(x) \le M \quad \forall x \in [x_{j-1},x_j], \quad j = 1, \dots, n$$

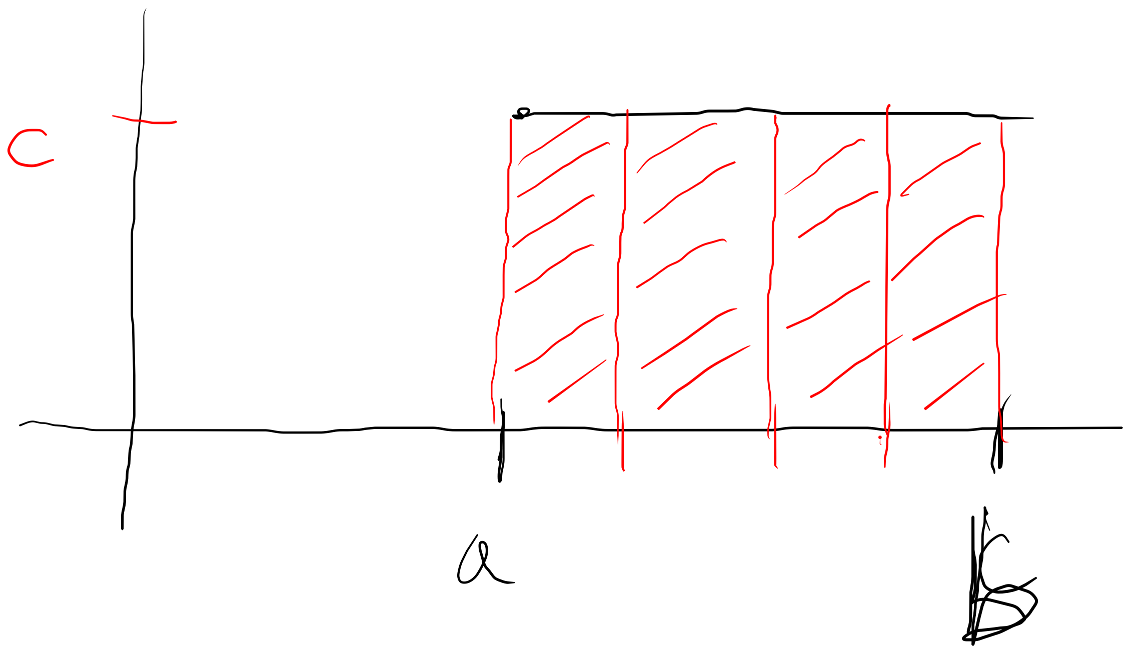
$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\sup f([x_{j-1},x_j])}_{\le M} \le \sum_{j=1}^{n} (x_j - x_{j-1}) M$$

$$= M \sum_{j=1}^{n} (x_j - x_{j-1}) = M(b-a)$$

$f(x) \equiv c \qquad m = M = c$

$$c(b-a) \leq s(\Delta) \leq S(\Delta) \leq c(b-a)$$

$$\Rightarrow \quad s(\Delta) = S(\Delta) = c(b-a)$$



$$D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$[a, b]$

$\Delta : x_0 < x_1 \ldots < x_n = b$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\sup D([x_{j-1}, x_j])}_{1} = \sum_{j=1}^{n} (x_j - x_{j-1}) = b - a$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \underbrace{\inf D([x_{j-1}, x_j])}_{0} = 0$$

Ricordiamo $D(\mathbb{R}) = \{0, 1\}$ e per ogni j

$$D([x_{j-1}, x_j]) = \{0, 1\}$$

$f : [a,b] \to \mathbb{R}$ crescente

(decrescente!)

$\Delta \qquad x_0 = a < x_1 < \cdots < x_n = b$

$f(x_{j-1})$

$$S(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup f([x_{j-1}, x_j]) = \sum_{j=1}^{n} (x_j - x_{j-1}) f(x_j)$$

$$s(\Delta) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf f([x_{j-1}, x_j]) = \sum_{j=1}^{n} (x_j - x_{j-1}) f(x_{j-1})$$

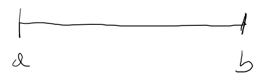
$f(x_j)$

Lemma: Data f limitata in $[a,b]$, siano $\Delta \succcurlyeq \Delta'$ due decomposizioni, con Δ' un raffinamento di Δ. Allora si ha

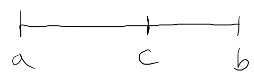
$$s(\Delta) \leq s(\Delta') \leq S(\Delta') \leq S(\Delta)$$

Dim: per un esempio

Δ $\quad x_0 = a < x_1 = b$



Δ' $\quad x_0 = a < x_1 = c < x_2 = b$



$$S(\Delta) \geq S(\Delta')$$

$$f([a,b]) \supseteq f([a,c]) \Rightarrow \sup f([a,b]) \geq \sup f([a,c])$$

$$\sup f([a,b]) \geq \sup f([c,b])$$

$$f([a,c]) = \{f(x) : x \in [a,c]\} \subseteq \{f(x) : x \in [a,b]\} = f([a,b])$$

$$S(\Delta) = (b-a) \sup f([a,b])$$

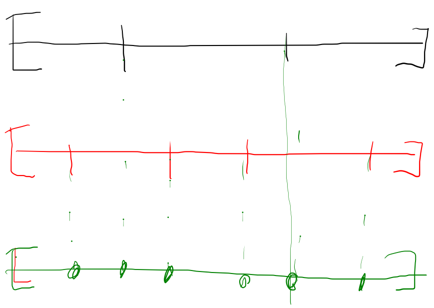
$$S(\Delta') = (c-a) \underbrace{\sup f([a,c])}_{\leq \sup f([a,b])} + (b-c) \underbrace{\sup f([c,b])}_{\leq \sup f([a,b])}$$

$$\leq (c-a) \sup f([a,b]) + (b-c) \sup f([a,b])$$

$$= \Big(\underbrace{(\cancel{c}-a) + (b-\cancel{c})}_{(b-a)} \Big) \sup f([a,b])$$

$$= (b-a) \sup f([a,b]) = S(\Delta)$$

Lemma Dato $[a,b]$ e due sue decomposizioni Δ, Δ' esiste una decomposizione Δ'' che è un raffinamento di entrambe.



Corollario Date due decomposizioni Δ, Δ', si ha $s(\Delta) \leq S(\Delta')$ $\ast$

Dim infatti per Δ'' che è un raffinamento di Δ e Δ'

$$s(\Delta) \leq s(\Delta'') \leq S(\Delta'') \leq S(\Delta')$$

$\Longrightarrow \ast$

In altre parole $\{s(\Delta)\}_{\Delta}$ $\{S(\Delta)\}_{\Delta}$ sono due classi separate

$$\underbrace{\sup \{s(\Delta)\}_{\Delta}}_{\underline{\int_a^b} f(x)\,dx} \leq \underbrace{\inf \{S(\Delta)\}_{\Delta}}_{\overline{\int_a^b} f(x)\,dx}$$