# Metrizzabilità dei proiettivi

Consideriamo le funzioni

$$\varphi_{ij} : \mathbb{R}P^n \longrightarrow \mathbb{R}$$
$$\varphi_{ij}([x_0, \dots, x_n]) = \frac{x_i x_j}{x_0^2 + \dots + x_n^2}$$

Sono ben definite e continue perché $\tilde{\varphi}_{ij} = \varphi_{ij} \circ \pi$

$$\tilde{\varphi}_{ij} : \mathbb{R}^{n+1} - \{0\} \longrightarrow \mathbb{R} \qquad \text{continua}$$
$$\tilde{\varphi}_{ij}(x_0, \dots, x_n) = \frac{x_i x_j}{x_0^2 + \dots + x_n^2}$$

Mostriamo che $\varphi_{ij}([x]) = \varphi_{ij}([y]) \ \forall i,j \implies [x] = [y]$

Non è restrittivo assumere $\|x\| = \|y\| = 1 \implies x_i x_j = y_i y_j \quad \forall i,j = 0, \dots, n$

$\implies x_i^2 = y_i^2 \ \forall i \implies x_i = \varepsilon_i y_i$ con $\varepsilon_i = \pm 1 \ \forall i$

$\exists\, i_0$ t.c. $x_{i_0} \neq 0 \implies y_{i_0} \neq 0 \implies \varepsilon_{i_0} y_{i_0} x_j = y_{i_0} y_j \quad \forall j = 0, \dots, n$

$\implies \varepsilon_{i_0} x_j = y_j \ \forall j = 0, \dots, n \implies y = \varepsilon_{i_0} x = \pm x \implies [x] = [y]$.

Ordiniamo le $\varphi_{ij} \rightsquigarrow \Psi : \mathbb{R}P^n \longrightarrow \mathbb{R}^{(n+1)^2}$ con componenti $\varphi_{ij}$

$\Psi$ continua e iniettiva, $\mathbb{R}P^n$ compatto, $\mathbb{R}^{(n+1)^2}$ di Hausdorff $\implies$

$\Psi$ immersione $\implies$ <u>$\mathbb{R}P^n$ metrizzabile</u> ($\implies$ <u>Hausdorff</u>).

Nel caso complesso si procede in modo simile ma le funzioni da considerare sono : $\quad \varphi_{ij} : \mathbb{C}P^n \longrightarrow \mathbb{C}$

$$\varphi_{ij}([x_0, \dots, x_n]) = \frac{x_i \bar{x}_j}{|x_0|^2 + \dots + |x_n|^2}$$

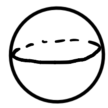
$\rightsquigarrow \Psi : \mathbb{C}P^n \longrightarrow \mathbb{C}^{(n+1)^2}$ immersione $\implies$ <u>$\mathbb{C}P^n$ metrizzabile</u> ($\implies$ <u>$T_2$</u>).

**Rette** $\mathbb{R}P^1 \rightsquigarrow H_0 = \{[0,1]\} \implies U_0 = \mathbb{R}P^1 - \{[0,1]\} \cong \mathbb{R} \implies$

$$\mathbb{R}P^1 \cong \hat{U}_0 \cong \hat{\mathbb{R}} \cong S^1 \qquad \boxed{\mathbb{R}P^1 \cong S^1} \qquad [0,1] = \infty.$$

$\mathbb{C}P^1 \rightsquigarrow H_0 = \{[0,1]\} \Rightarrow U_0 = \mathbb{C}P^1 - \{[0,1]\} \cong \mathbb{C} \Rightarrow$

$\mathbb{C}P^1 \cong \hat{U}_0 \cong \hat{\mathbb{C}} \cong \hat{\mathbb{R}}^2 \cong S^2$ $\qquad \boxed{\mathbb{C}P^1 \cong S^2}$ $\qquad [0,1] = \infty .$

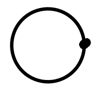


<u>NB</u> $\mathbb{R}P^n$ e $\mathbb{C}P^n$ <u>non</u> sono omeomorfi a sfere per $n > 1$.

## Incollamenti topologici

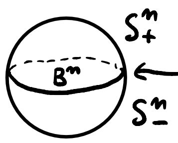
- $A \subset X$, $\varphi : A \to X \rightsquigarrow \underline{X_\varphi := X / (a \sim f(a) \ \forall a \in A)}$
  $X_f$ ottenuto da $X$ incollando ciascun punto $a \in A$ con $\varphi(a)$ con la topologia quoziente.

- $A \subset X$, $\varphi : A \to Y \rightsquigarrow \underline{X \cup_\varphi Y := (X \sqcup Y)/(a \sim \varphi(a) \ \forall a \in A)}$
  unione di $X$ e $Y$ mediante $\varphi$ con la topologia quoziente.
  Se la mappa $\varphi$ è ovvia si scrive anche $X \cup_A Y$.

<u>Es</u> $\quad \varphi : \{0\} \to [0,1]$, $\varphi(0) = 1 \rightsquigarrow [0,1]/(0 \sim 1) \cong S^1$ 

## Sfere

$S^n = S^n_+ \cup S^n_- \qquad S^n_+ = \{(x_0, \dots, x_n) \mid x_n \geq 0\}$

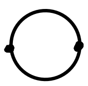


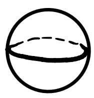

$S^n_+ \cap S^n_- = S^{n-1} \qquad S^n_- = \{(x_0, \dots, x_n) \mid x_n \leq 0\}$

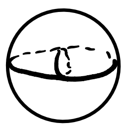
$\pi_+ : S^n_+ \xrightarrow{\cong} B^n$, $\pi_+(x_0, \dots, x_n) = (x_0, \dots, x_{n-1})$ omeo

$\pi_- : S^n_- \to B^n$, $\pi_-(x_0, \dots, x_n) = (x_0, \dots, x_{n-1})$ omeo

$\Rightarrow S^n \cong B^n \cup_{S^{n-1}} B^n$, $\partial B^n = S^{n-1}$ incollato mediante $id_{S^{n-1}}$

<u>Es</u> $\quad S^1 = B^1 \cup_{\{0,1\}} B^1$ 

$S^2 = B^2 \cup_{S^1} B^2$ 

$S^3 = B^3 \cup_{S^2} B^3$ 

# $\mathbb{R}P^n$

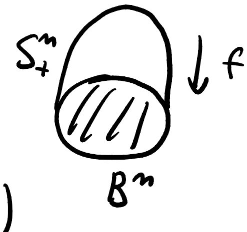
$p: S^n \to \mathbb{R}P^n$, $p(x) = [x]$ ⇝ $S^n_+ = \{(x_0, \dots, x_n) \in S^n \mid x_n \geq 0\}$

$p|_{S^n_+}: S^n_+ \to \mathbb{R}P^n$ suriettiva e $\forall x, y \in S^n_+$,

$p(x) = p(y) \iff x = y$ o $x = -y \in S^{n-1} \subset S^n \Rightarrow$

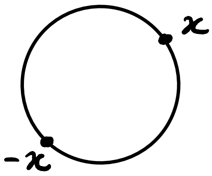
$\mathbb{R}P^n = S^n_+/(x \sim -x \ \forall x \in S^{n-1} \subset S^n_+)$


$f: S^n_+ \xrightarrow{\cong} B^n$, $f(x_0, \dots, x_n) = (x_0, \dots, x_{n-1})$ omeo




$f|_{S^{n-1}} = id_{S^{n-1}} \Rightarrow \mathbb{R}P^n \cong B^n/(x \sim -x \ \forall x \in S^{n-1})$

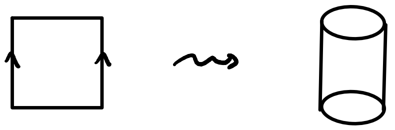
e l'immagine di $S^{n-1}$ è $\mathbb{R}P^{n-1} \subset \mathbb{R}P^n$.

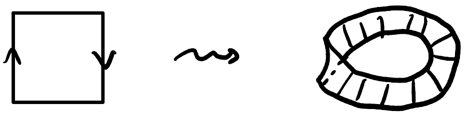


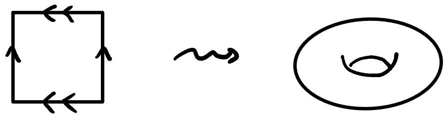

**Es** $\mathbb{R}P^2 \cong B^2/(x \sim -x \ \forall x \in S^1)$

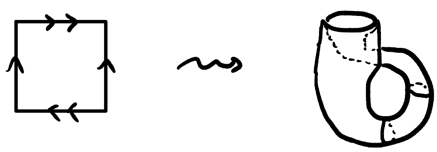
## Quozienti notevoli del quadrato

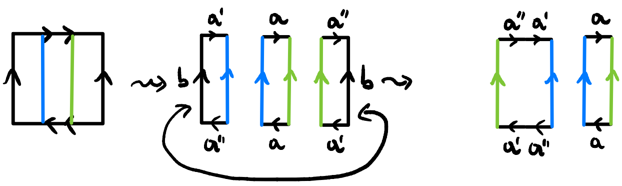
Identifichiamo linearmente i lati in accordo con le frecce.

1)  $\cong S^1 \times I$ cilindro

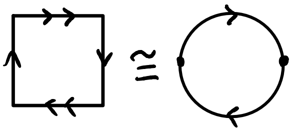
2)  Striscia di Möbius $Mb$, $\partial(Mb) \cong S^1$

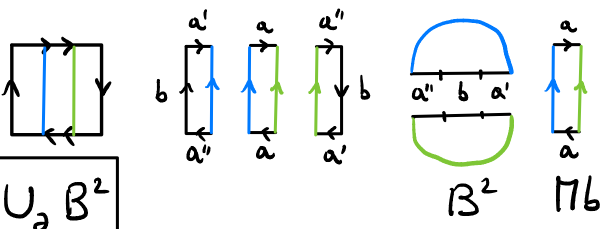
3)  $\cong T^2$ toro

4)  Bottiglia di Klein $Kl$




$\cong Mb \sqcup Mb \Rightarrow Kl \cong Mb \cup_{S^1} Mb$

5)  $\cong \mathbb{R}P^2$



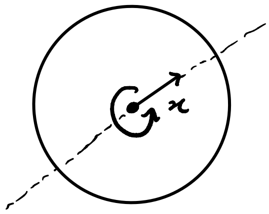

$$\boxed{\mathbb{R}P^2 \cong Mb \cup_\partial B^2}$$

# Relazione d'equivalenza indotta da un'applicazione

$f : X \to Y \rightsquigarrow x_1 \sim_f x_2 \iff f(x_1) = f(x_2)$ su $X \rightsquigarrow X/\sim_f$ .

## $RP^3 \cong SO(3)$

$\varphi : B^3 \to SO(3)$ suriettiva e continua




$x \neq 0$, $\varphi(x) =$ rotazione di asse $x$ e angolo $\pi \|x\|$
in senso antiorario

$\varphi(0) = id_{R^3}$

$\varphi(x) = \varphi(y) \iff x = y$ o $x = -y \in S^2 \implies B^3/\sim_\varphi \cong RP^3$

$$\begin{tikzcd}
B^3 \arrow[r, "\varphi"] \arrow[d, "\pi"'] & SO(3) \\
RP^3 \cong B^3/\sim_\varphi \arrow[ur, "\cong", "\bar{\varphi}"'] &
\end{tikzcd}
\qquad \bar{\varphi}([x]) = \varphi(x)$$

$\bar{\varphi}$ ben definita e continua perché $\pi \circ \bar{\varphi} = \varphi$ continua

$\bar{\varphi}$ biiettiva, $B^3/\sim_\varphi$ compatto, $SO(3)$ $T_2$ $\implies \tilde{\varphi}$ omeo

$\implies RP^3 \cong SO(3)$.