# Lezione 28 novembre 22

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**VISTO** : Rette piane affini.

## POSIZIONI REC. TRA SSA

**Def.**: Siano $S, S' \subseteq A^n_R$ due sottospazi affini. Allora $S$ e $S'$ si dicono:

① **INCIDENTI** se $S \cap S' \neq \emptyset$

(e **COINCIDENTI** se $S = S'$)

② **PARALLELI** e scriveremo $S \parallel S'$ se

$$W_S \subseteq W_{S'} \quad \text{opp.} \quad W_{S'} \subseteq W_S$$

dove $W_S$ = giacitura di $S$

$W_{S'}$ = giacitura di $S'$

③ **SGHEMBI** se NON sono INCIDENTI NÈ PARALLELI

## GEOMETRIA AFF. DI $A^3_R$ $(= \mathbb{R}^3)$

$\dim A^3_{\mathbb{R}} = 3$ ; $S \subseteq A^3_{\mathbb{R}}$

$\dim S =$
- $0$ $S = \{Q\}$
- $1$ retta
- $2$ piano
- $3$ : $S = A^3_{\mathbb{R}}$

## RETTE DI $A^3_R$:

$r$ : retta $Q = (q_1, q_2, q_3) \in A^3$

$$W_r = \mathrm{Span}\left(\begin{pmatrix} l \\ m \\ n \end{pmatrix}\right) \subseteq \mathbb{R}^3$$

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$r: \begin{cases} x = q_1 + l t \\ y = q_2 + m t \\ z = q_3 + n t \end{cases} \qquad \begin{cases} x - q_1 = l t \\ y - q_2 = m t \\ z - q_3 = n t \end{cases}$$

passaggio a eq. cartesiane:

$$\mathrm{rg}\begin{pmatrix} l & x - q_1 \\ m & y - q_2 \\ n & z - q_3 \end{pmatrix} = 1$$

mettendo a scala, e annullando le ultime due righe, troviamo 2 eq. cartesiane.

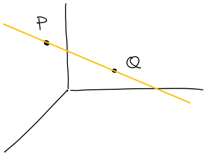
Viceversa: se $r: \begin{cases} a_1 x + b_1 y + c_1 z = d_1 \\ a_2 x + b_2 y + c_2 z = d_2 \end{cases}$

Trovo eq. parametriche per r risolvendo il SL e scrivendo la sua generica sol. in funzione di 1 parametro.

RETTE PER 2 PUNTI : $\exists !$

$Q = (q_1, q_2, q_3)$, $P = (p_1, p_2, p_3)$

$Q \neq P$



Scelgo Q come punto base; $P \in r$

$\Longleftrightarrow \overrightarrow{QP} \in W_r$, $\dim W_r = 1$

$\Rightarrow \dim W_r = \mathrm{Span}(\overrightarrow{QP})$

$$\overrightarrow{QP} = \begin{pmatrix} p_1 - q_1 \\ p_2 - q_2 \\ p_3 - q_3 \end{pmatrix} = \begin{pmatrix} \ell \\ m \\ n \end{pmatrix}$$

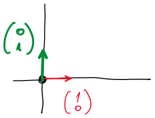
$$\Rightarrow \begin{cases} x = q_1 + \overset{=\ell}{\boxed{(p_1 - q_1)}}\, t \\ y = q_2 + \overset{=m}{\boxed{(p_2 - q_2)}}\, t \\ z = q_3 + \overset{=n}{\boxed{(p_3 - q_3)}}\, t \end{cases}$$

Esempio: assi coordinati in $\mathbb{A}^3_{\mathbb{R}}$

Ricordiamo (dai tempi scolastici) la situazione in $\mathbb{A}^2_{\mathbb{R}}$:

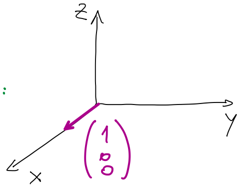
eq. parametriche

: Asse x : $\underset{\text{eq. cartesiana}}{y=0} \Rightarrow \begin{cases} x = t \\ y = 0 \end{cases}$ : $Q = (0,0)$, $\begin{pmatrix} \ell \\ m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$


$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Asse y: $x = 0 \Rightarrow \begin{cases} x = 0 \\ y = t \end{cases}$ : $Q = (0,0)$, $\begin{pmatrix} \ell \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

In $\mathbb{A}^3_{\mathbb{R}}$ la situazione è simile:

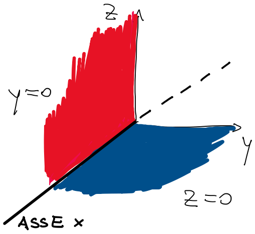


Asse x : $Q = (0,0,0)$, $\begin{pmatrix} \ell \\ m \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \quad , \quad \text{ovvero} \quad \begin{pmatrix} 1 & x-0 \\ 0 & y-0 \\ 0 & z-0 \end{pmatrix}$$

$$\text{rg} \begin{pmatrix} \textcircled{1} & x \\ 0 & y \\ 0 & z \end{pmatrix} = 1$$

$\iff$ $2^a$ e $3^a$ riga nulle: $\begin{cases} y=0 \\ z=0 \end{cases}$



Analog: Asse y: $\begin{cases} x = 0 \\ y = t \\ z = 0 \end{cases}$ eq. cart. $\begin{cases} x=0 & : \text{piano } yz \\ z=0 & : \text{piano } xy \end{cases}$

$Q=(0,0,0)$

$$\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Asse z: $\begin{cases} x = 0 \\ y = 0 \\ z = t \end{cases}$ , eq. cart. $\begin{cases} x=0 \\ y=0 \end{cases}$

Esempio: retta $Q=(4,3,7)$, $\begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$

$$\begin{cases} x = 4 + 1 \cdot t \\ y = 3 + 3 \cdot t \\ z = 7 + 5t \end{cases}$$

eq. cart.: $\text{rg} \begin{pmatrix} 1 & x-4 \\ \textcircled{3} & y-3 \\ \textcircled{5} & z-7 \end{pmatrix} = 1$

$$\begin{array}{l} \underline{II} - 3 \cdot I \\ \rightsquigarrow \\ \underline{III} - 5 \cdot I \end{array} \quad \begin{pmatrix} 1 & x-4 \\ 0 & y-3-3(x-4) \\ 0 & z-7-5 \cdot (x-4) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & x-4 \\ 0 & -3x+y+9 \\ 0 & -5x+z+13 \end{pmatrix}$$

$$\operatorname{rg} = 1 \iff \begin{cases} -3x+y = -9 & /\cdot(-1) \\ -5x+z = -13 & /\cdot(-1) \end{cases}$$

$$\begin{cases} 3x-y = 9 \\ 5x-z = 13 \end{cases}$$

## PIANI in $\mathbb{A}^3_{\mathbb{R}}$

$S \subseteq \mathbb{A}^3_{\mathbb{R}}$ ; $\dim S = \dim W_S = 2$

$Q = (q_1, q_2, q_3)$, $W_S = \operatorname{Span}\left( \begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}, \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} \right)$

con $\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix}$ non propor. a $\begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix}$

eq. par.
$$\boxed{\begin{cases} x-q_1 = l_1 \cdot t + l_2 \cdot s \\ y-q_2 = m_1 \cdot t + m_2 \cdot s \\ z-q_3 = n_1 \cdot t + n_2 \cdot s \end{cases}}$$

eq. cart.:
$$\operatorname{rg} \underbrace{\begin{pmatrix} l_1 & l_2 & x-q_1 \\ m_1 & m_2 & y-q_2 \\ n_1 & n_2 & z-q_3 \end{pmatrix}}_{\downarrow} = 2$$

$$\iff \det \qquad = 0$$

Viceversa: $ax+by+cz = d$

risolvo l'eq: $a \neq 0$ :
$z = s$
$y = t$
$x = (d-bt-cs)/a$

cioè:
$$\begin{cases} x = \frac{d}{a} - \frac{b}{a}t - \frac{c}{a}s \\ y = t \\ z = s \end{cases}$$

$Q = \left(d/a, 0, 0\right)$

$$\begin{pmatrix} l_1 \\ m_1 \\ n_1 \end{pmatrix} = \begin{pmatrix} -\frac{b}{a} \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} l_2 \\ m_2 \\ n_2 \end{pmatrix} = \begin{pmatrix} -\frac{c}{a} \\ 0 \\ 1 \end{pmatrix}$$

ovvero: $\operatorname{Span}\left(\begin{pmatrix} -b/a \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -c/a \\ 0 \\ 1 \end{pmatrix}\right)$

$$= \operatorname{Span}\left(\begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}, \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}\right)$$

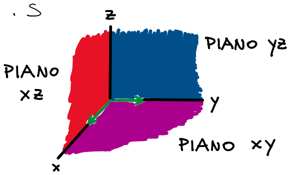
Esempio : piani coordinati

Piano $xy$ : piano per $(0,0,0)$ e

$$W_{xy} = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$$

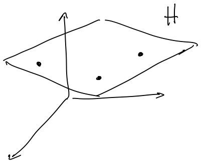
$$\begin{cases} x = 0 + 1 \cdot t + 0 \cdot s \\ y = 0 + 0 \cdot t + 1 \cdot s \\ z = 0 + 0 \cdot t + 0 \cdot s \end{cases}$$

$$\begin{cases} x = t \\ y = s \\ z = 0 \end{cases}$$



$$\underbrace{\det\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & z \end{pmatrix}}_{1 \cdot z \,=\, 0} = 0 \qquad z = 0$$

## PIANI PER 3 punti DISTINTI & NON ALLINEATI : ∃!



$Q = (q_1, q_2, q_3)$

$P = (p_1, p_2, p_3)$

$R = (r_1, r_2, r_3)$

Scelgo $Q$ come punto base

$P \in H \Rightarrow \overrightarrow{QP} \in W_H$

$R \in H \Rightarrow \overrightarrow{QR} \in W_H$

$P, Q, R$ non allineati $\Rightarrow \overrightarrow{QP}$ e $\overrightarrow{QR}$

sono LIN. INDIP.

$$\Rightarrow W_H = \operatorname{Span}\left(\overrightarrow{QP}, \overrightarrow{QR}\right)$$

$$\Rightarrow \begin{cases} x = q_1 + (p_1 - q_1)\cdot t + (r_1 - q_1)\cdot s \\ y = q_2 + (p_2 - q_2)\cdot t + (r_2 - q_2)\cdot s \\ z = q_3 + (p_3 - q_3)\cdot t + (r_3 - q_3)\cdot s \end{cases}$$

ALTRO MODO: per trovare eq. cart. di H

$$ax + by + cz = d$$

$$\begin{cases} a\cdot q_1 + b q_2 + c q_3 - d = 0 & \text{pass. per } Q \\ a\cdot p_1 + b\cdot p_2 + c\cdot p_3 - d = 0 & \text{pass. per } P \\ a r_1 + b r_2 + c r_3 - d = 0 & \text{pass. per } R \end{cases}$$

SLO nelle inc. $a, b, c, d$

$$\overrightarrow{QP} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Esempio: $Q = (1, 0, 1)$ $\quad r: \begin{cases} x = 1 - t \\ y = 0 \\ z = 1 \end{cases}$ retta per P e Q

$P = (0, 0, 1)$

$R = (2, 1, 0)$ $\quad : R \overset{?}{\in} r$

Sono allineati? $\quad \begin{cases} 2 = 1 - t \\ 1 = 0 \\ 0 = 1 \end{cases}$ SL INCOMP.

$\Rightarrow R \notin r$, non sono allineati

$$\begin{array}{l} \text{pass per } Q: \\ P: \\ R: \end{array} \begin{cases} a\cdot 1 + b\cdot 0 + c\cdot 1 - d = 0 \\ a\cdot 0 + b\cdot 0 + c\cdot 1 - d = 0 \\ a\cdot 2 + b\cdot 1 + c\cdot 0 - d = 0 \end{cases} \qquad \begin{cases} a + c - d = 0 \Rightarrow a = 0 \\ c - d = 0 \Rightarrow c = d \\ 2a + b - d = 0 \Rightarrow b = d \end{cases}$$

$d = 1$ : $\quad b = 1, \quad c = 1, \quad a = 0$ :

$$\boxed{y + z = 1}$$

$$d\cdot y + d\cdot z = d, \quad dy + dz - d = 0$$

$$d\cdot(y + z - 1) = 0$$