Thm Let $S \subset H$ be orthonormal

1) For any $u \in H$

$$\sum_{s \in S} |(u, s)_H|^2 \leq |u|_H^2$$ Bessel inequality

2) $V_S = \overline{\operatorname{sp}\{S\}}$. The following are equiv.

a) $u \in V_S$

b) $\sum_{s \in S} |(u, s)_H|^2 = |u|_H^2$ Parseval identity

c) $\sum_{s \in S} (u, s)_H \, s = u$ in H

3) $\forall u \in H$ the series $\sum_{s \in S} (u, s)_H \, s = P_{V_s} u$

$$\sum_{s \in S} |(u, s)_H|^2 = |P_{V_s} u|_H^2$$ Parseval identity.

Pf Assume S countable, $u \in H$

$\{s_j\}_{j \in \mathbb{N}}$ $s_1, \dots, s_n$

$$S_n u = \sum_{j=1}^{n} (u, s_j)_H \, s_j$$

$$|S_n u|_H^2 = \sum_{j=1}^{n} |(u, s_j)_H|^2$$

$$|u - S_n u|_H^2 = |u|_H^2 - |S_n u|_H^2$$

$$(u - S_n u, u - S_n u)_H = |u|_H^2 + |S_n u|_H^2 - 2\,\mathrm{Re}(u, S_n u)$$

$$\mathrm{Re}(u, S_n u)_H = |S_n u|_H^2$$

$$(u, S_n u) = (S_n u, S_n u) + \underbrace{(u - S_n u, S_n u)}_{0}$$

$$(u - S_n u, S_n u) = \Big(u - \sum_{k=1}^{n} (u, s_k) s_k, \sum_{j=1}^{n} (u, s_j) s_j\Big)$$

$$= \sum_{j=1}^{n} (u, s_j)\overline{(u, s_j)}$$

$$- \sum_{k=1}^{n} \sum_{j=1}^{n} (u, s_k) \underbrace{(s_k, s_j)}_{\delta_{kj}} \overline{(u, s_j)} = 0$$

$$(u - S_n u, u - S_n u)_H = |u|_H^2 + |S_n u|_H^2 - 2\,\mathrm{Re}(u, S_n u)$$

$$= |u|_H^2 + |S_n u|_H^2 - 2\,|S_n u|_H^2$$

$$0 \leq |u - S_n u|_H^2 = |u|_H^2 - |S_n u|_H^2$$ ✓

$$|u|_H^2 \geq |S_n u|_H^2 = \sum_{j=1}^{n} |(u, s_j)|^2$$

$$|u|_H^2 \geq \sum_{j=1}^{\infty} |(u, s_j)_H|^2 = \sum_{s \in S} |(u, s)_H|^2$$

$$|u|_H^2 \geq \sum_{s \in S} |(u, s)_H|^2$$

Only for on at most countable subset of S we can have $(u, s)_H \neq 0$

2) $V = \overline{sp\{S\}}$. The following are equiv.

a) $u \in V_S$

b) $\sum_{s \in S} |(u, s)_H|^2 = |u|_H^2$ Parseval identity

c) $\sum_{s \in S} (u, s)_H \, s = u$ in H

$a \Rightarrow c$

If $u \in V$ $\forall \varepsilon > 0$ $\exists$ $s_1, \ldots, s_n \in S$

$\lambda_1, \dots, \lambda_n$ st $\left| u - \sum_{j=1}^{n} \lambda_j s_j \right|_H < \varepsilon$

Claim

$$\Big| u - \underbrace{\sum_{j=1}^{n} (u, s_j)_H s_j}_{S_n u} \Big|_H \leq \Big| u - \sum_{j=1}^{n} \lambda_j s_j \Big|_H < \varepsilon$$

$$\Longrightarrow \qquad \lim_{n \to +\infty} S_n u = u \Rightarrow c)$$

$$u = \sum_{s \in S} (u, s)_H s$$

$$\Big| u - \sum_{j=1}^{n} \lambda_j s_j \Big|_H^2 = \Big| u - S_n u + S_n u - \sum_{j=1}^{n} \lambda_j s_j \Big|_H^2$$

$$= \Big| \Big(u - \sum_{j=1}^{n} (u, s_j)_H s_j \Big) + \Big(\sum_{j=1}^{n} \big((u, s_j)_H - \lambda_j \big) s_j \Big) \Big|_H^2$$

$$= \Big| u - \sum_{j=1}^{n} (u, s_j)_H s_j \Big|_H^2 + \underbrace{\sum_{j=1}^{n} \big| (u, s_j)_H - \lambda_j \big|^2}_{\geq 0}$$

$$\geq \big| u - S_n u \big|_H^2$$

$a \Longrightarrow c$

$a \Longleftarrow c$

$$\sum_{s \in S} (u, s) s = u \quad \text{in } H \Longrightarrow u \in V = \overline{\mathrm{Sp}(S)}$$

Assume c) $\Longrightarrow$ b)

$$S_n u \longrightarrow u \quad \text{in} \quad H$$

$$L = \sum_{j=1}^{n} (u, s_j) s_j$$

$$|S_n u|_H^2 \longrightarrow |\cancel{S}u|_H^2$$

$$\sum_{j=1}^{n} |(s_j, u)|^2 \xrightarrow{n \to +\infty} |u|_H^2$$

$$\Longrightarrow \quad \sum_{s \in S} |(s, u)_H|^2 = |u|_H^2$$

b $\Rightarrow$ c

$$|u - S_n u|_H^2 = |u|_H^2 - |S_n u|_H^2 =$$

$$= |u|_H^2 - \sum_{j=1}^{n} |(s_j, u)|^2 \xrightarrow{n \to +\infty} 0$$

$$\Longrightarrow \quad \underline{S_n u \longrightarrow u \quad \text{in} \quad H}$$

$u \in H$ $$\sum_{s \in S} (u, s) s = P_V u$$

$P_V : H \longrightarrow V$ $$\sum_{s \in S} |(u, s)|^2 = |P_V u|_H^2$$

$$P_V u \in V \Rightarrow \sum_{s \in S} (P_V u, s) s = P_V u$$

$$\forall s, \quad (P_V u, s)_H = (u, s)_H$$

by $\cancel{Re}(u - P_V u, v - \cancel{P_V} u)_H \leq 0 \quad \forall v \in V$

$$(u - P_V u, v) = 0 \qquad \forall v \in V$$

$$\Rightarrow \sum_{s \in S} (u, s) s = P_V u$$

$$\sum_{s \in S} |(P_V u, s)_H|^2 = |P_V u|_H^2$$

$$\Leftrightarrow \sum_{s \in S} |(u, s)|_H^2 = |P_V u|_H^2$$

$$L^2(\mathbb{T}^d) \qquad \mathbb{T} = \mathbb{R} / 2\pi \mathbb{Z} \qquad d = 1$$

$$S = \left\{ \frac{e^{inx}}{\sqrt{2\pi}} : n \in \mathbb{Z} \right\}$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} e^{inx} e^{-imx} \, dx = \delta_{nm}$$

Is a Hilbert basis for $L^2(\mathbb{T})$

$V = \overline{sp(S)}$ in $L^2(\mathbb{T})$

V is the space of trigonometric polynomials

The closure of V in $C^0(\mathbb{T})$

is $C^0(\mathbb{T})$ $\qquad \sigma_n f \longrightarrow f$

$\overline{C^0(\mathbb{T})}$ in $L^2(\mathbb{T})$ is the whole $L^2(\mathbb{T})$

$$C^0(\mathbb{T}) \hookrightarrow L^2(\mathbb{T})$$

$$\|\cdot\|_\infty \qquad \|\cdot\|_{L^2}$$

$$\sum_{n\in\mathbb{Z}} \left| \left(u, \frac{e^{inx}}{\sqrt{2\pi}} \right) \right|^2 = |u|^2_{L^2(\mathbb{T})}$$

$$= \sum_{n\in\mathbb{Z}} \left| \frac{\overset{\sqrt{2\pi}}{1}}{2\pi\cancel{\sqrt{2\pi}}} \int_{-\pi}^{\pi} e^{-inx} u(x)\,dx \right|^2$$

$$= \sum_{n\in\mathbb{Z}} \left| \sqrt{2\pi} \quad \hat{u}(n) \right|^2 = 2\pi \sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2$$

$$= |u|^2_{L^2(\mathbb{T})}$$

$$L^2(\mathbb{T}) \longrightarrow \ell^2(\mathbb{Z})$$

$$U \longrightarrow \sqrt{2\pi}\,\hat{U}(n)$$

__Def__ H $T \in \mathcal{L}(H)$, T^* remains defined by

$$(Tx, y)_H = (x, T^* y)_H \quad \forall x, y \in H$$
$$(x, Ty)_H = (T^* x, y)$$

If $T = T^*$ then T is ~~self~~ self-adjoint

T is a ~~unitary~~ unitary operator if it is an isometric isomorfism.

__Def__ $T \in \mathcal{L}(H)$ is positive if

$$(Tx, x)_H \geq 0 \quad \forall x \in H$$

$T \geq 0$

__Remark__ If $T \geq 0$ and H is a Hilbert space on $\mathbb{C}$ then T is ~~self~~ selfadjoint.

__Lemma__ For $T \in \mathcal{L}(H)$ self adjoint operator

the $H = \ker T \oplus \ker^{\perp} T$

is invariant under the action of T on

$$\ker^{\perp} T = \overline{R(T)}$$

Pf $\quad H = V \oplus V^{\perp}$

$$P_V : H \to V$$

$$|x|_H^2 = |P_V x|_H^2 + |\underbrace{x - P_V x}|_H^2$$

$$x = P_V x + (1 - P_V)x$$

$$T : \ker T \supseteq 0$$

$$x \in \ker T \qquad Tx = 0 \in \ker T$$

$$T : \ker^{\perp} T \supseteq$$

$$y \in \ker^{\perp} T \qquad (x, y)_H = 0 \quad \forall\, x \in \ker T$$

$$(T^*)^* = T$$

$$(x, Ty)_H =$$

$$= (T^* x, y)_H = (Tx, y)_H = (0, y)_H = 0$$

$$\ker^{\perp} T = \overline{R(T)}$$

$$(x, Ty)_H = (Tx, y)_H \qquad \forall\, x, y \in H$$

$$x \in \ker T \implies (x, Ty)_H = 0 \qquad \forall\, y \in H$$

$$\implies \overline{R(T)} \subseteq \ker^{\perp} T$$

If $\quad \ker^{\perp} T \underset{\neq}{\supset} \overline{R(T)}$

$\Rightarrow \quad z \in \ker^{\perp}(T) \setminus \overline{R(T)}$

$z \in \overline{R(T)}^{\perp}$

$\Rightarrow 0 = (\cancel{z}, Ty)_H = (Tz, y)_H = 0 \quad \forall y$

$\forall y$

$Tz = 0 \Rightarrow \overset{0}{\cancel{z}} \in \ker T$

$z \in \ker^{\perp} T$

$|z|_H^2 = (z, z)_H = 0$

T selfadjoint

$H = \ker T \oplus \ker^{\perp} T$

$T \in \mathcal{L}(H)$

$H = N_g(T) \oplus N_g^{\perp}(T)$

$N_g(T) = \overline{\bigcup_{j=1}^{\infty} \ker T^j}$

__Theorem__ Let $A \in \mathcal{L}(H)$ $A \geq 0$ and $A = A^*$.

Then there exists a unique selfadjoint positive and bounded operator $\sqrt{A} = A^{\frac{1}{2}}$, s.t.

$$\left(A^{\frac{1}{2}}\right)^2 = A.$$

Theorem For $A \in \mathcal{L}(H)$ there exists U unitary and R positive self adjoint s.t. $A = UR$.

R is unique, we can can write it in the form $R = |A|$, and is defined as $\underbrace{\sqrt{A^* A}}_{R}$

$f(A)$

$U = f(A)$ $\qquad f(t) = \frac{t}{|t|}$