Dec 7.

$$|(a,b)_H| \leq \|a\|_H \|b\|_H$$

$$|\mathrm{Re}\,(a,b)_H| \leq \|a\|_H \|b\|_H$$

$$\|a+b\|_H \leq \|a\|_H + \|b\|_H \quad \checkmark$$

$$\|a+b\|_H^2 = (a+b, a+b)_H =$$

$$= \|a\|_H^2 + \|b\|_H^2 + 2\,\mathrm{Re}\,(a,b)_H$$

$$\leq \|a\|_H^2 + \|b\|_H^2 + 2\|a\|_H \|b\|_H$$

$$= \left(\|a\|_H + \|b\|_H\right)^2$$

<u>Prop</u> H Hilbert $\Rightarrow$ H unif convex.

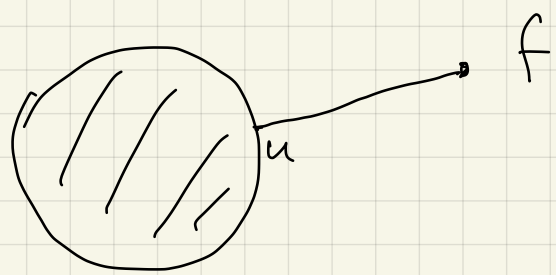
Thm K a closed convex subset of H

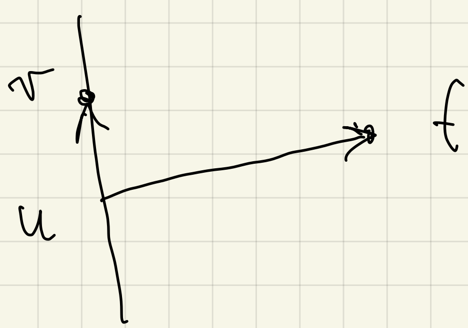
Then $\forall\ f \in H \quad \exists!\ u \in K$

s.t. $\quad \| f - u \|_H \leq \| f - v \|_H \quad \forall\ v \in K$

u is characterized by

$$\mathrm{Re}\,(f - u, v - u)_H \leq 0 \qquad \forall\ v \in K$$





Pf $\quad \phi(x) = \| x - f \|_H \qquad x \in K$

$\phi \in C^0(K, \mathbb{R})$, $\quad \phi$ is convex

$\lim\limits_{x \to \infty} \phi(x) = +\infty \qquad$ then ϕ has an absolute minimum.

$\cancel{\exists}$ $\exists$ $\{x_n\}$ in K s.t.

$$d_n = \| x_n - f \|_H$$

$$d_n \xrightarrow{n \to +\infty} d \doteq \inf_{x \in K} \| x - f \|_{\cancel{K}H}$$

$$\left\| \frac{a+b}{2} \right\|_H^2 + \left\| \frac{a-b}{2} \right\|_H^2 = \frac{1}{2} \left(\| a \|_H^2 + \| b \|_H^2 \right)$$

$a = f - x_n$

$b = f - x_m$

$$\left\| f - \frac{x_n + x_m}{2} \right\|_H^2 + \left\| \frac{x_n - x_m}{2} \right\|_H^2 = \frac{1}{2} \left(d_n^2 + d_m^2 \right)$$

$\cancel{K}$

$$\left\| \frac{x_n - x_m}{2} \right\|_H^2 = \frac{1}{2} \left(d_n^2 + d_m^2 \right) - \left\| f - \left(\frac{x_n + x_m}{2} \right) \right\|_H^2$$

$$\le \frac{1}{2} \left(d_n^2 + d_m^2 \right) - d^2$$

$$d_n^2 \to d^2 \qquad d_m^2 \to d^2$$

$$\lim_{(m,n)\to(\infty,\infty)} \|x_n - x_m\|_H = 0$$

$\Rightarrow$ $\{x_n\}$ is Cauchy in K

$$x_n \longrightarrow u \in K$$

$$d = \|u - f\|_H$$

a) u is a minimizer of $\phi(x) = |x - f|_H$

b) $\mathrm{Re}(f-u, v-u)_H \le 0 \quad \forall v \in K$

Assume for a moment the equivalence and suppose u_1, u_2 are minimum

$$\mathrm{Re}(f-u_1, v-u_1)_H \le 0 \qquad \forall v$$

$$\mathrm{Re}(f-u_2, v-u_2)_H \le 0 \qquad \forall v$$

$$0 \ge \mathrm{R}(f-u_1, u_2-u_1) + \mathrm{Re}(f-u_2, u_1-u_2)$$

$$= \mathrm{Re}(f-u_1, u_2-u_1) - \mathrm{Re}(f-u_2, u_2-u_1)$$

$$= \mathrm{Re}(f-u_2+u_2-u_1, u_2-u_1) - \mathrm{Re}(f-u_2, u_2-u_1)$$

$$= \cancel{\mathrm{Re}(f-u_2, u_2-u_1)} + |u_2-u_1|_H^2 - \cancel{\mathrm{Re}(f-u_2, u_2-u_1)}$$

$$0 \geq |u_2-u_1|_H^2 \Rightarrow u_2 = u_1$$

a) u is a minimyer of $\phi(x) = |x-f|_H$

b) $\mathrm{Re}(f-u, v-u)_H \leq 0 \quad \forall v \in K$

Assume a) and let $v \in K$ $\quad t \in [0,1]$

$$\|f-u-t(v-u)\|_H^2 = \|f-((1-t)u+tv)\|_H^2.$$

$$= \|f-u\|_H^2 - 2t\,\mathrm{Re}(f-u, v-u) + t^2 \|v-u|_H^2$$

$t=0$ is a point of minimi

$$\frac{d}{dt} \|f-u-t(v-u)\|_H^2 \Big|_{t=0} = -2\,\mathrm{Re}(f-u, v-u) \geq 0$$

$$\Rightarrow \qquad \operatorname{Re}(f-u, v-u) \le 0 \quad \forall v \in K$$

so $\quad a \Rightarrow b$

Now assume (b). We have a point u where

$$\operatorname{Re}(f-u, v-u) \le 0 \quad \forall v \in K.$$

$$|u-f|_H^2 - |v-f|_H^2$$

$$= |u|_H^2 + \cancel{|f|_H^2} - 2\operatorname{Re}(u,f)$$

$$- |v|_H^2 - \cancel{|f|_H^2} + 2\operatorname{Re}(v,f)$$

$$= |u|_H^2 - |v|_H^2 + 2\operatorname{R}(v-u, f-u+u)$$

$$= 2\operatorname{Re}(v-u, f-u)_H + 2\operatorname{Re}(v-u, u)_H + |u|_H^2 - |v|_H^2$$

$$= 2\operatorname{Re}(v-u, f-u)_H + 2\operatorname{Re}(v, u)_H - |u|_H^2 - |v|_H^2$$

$$= \underbrace{2\operatorname{Re}(v-u, f-u)_H}_{\le 0} - |u-v|_H^2 \le -|u-v|_H^2$$

$$\mathrm{Re}\,(f-u, v-u) \leq 0 \quad \forall v \in K. \quad (b)$$

$$|u-f|_H^2 - |v-f|_H^2 \leq -|u-v|_H^2$$

$\forall v \in K$

$\Rightarrow$ u is the unique minimizer

<u>Prop</u> K closed or convex in H and if $\forall f \in H$ I denote by $P_K f$ the u, then P_K is a contraction $P_K : H \to K$

$$|P_K f - P_K g|_H \leq |f-g|_H$$

<u>Pf</u> $u = P_K f$, $v = P_K g$

$$\mathrm{Re}\,(f-u,\, w-u) \leq 0 \qquad \forall w \in K$$

$$\mathrm{Re}\,(g-v,\, w-v) \leq 0 \qquad \forall w \in K$$

$$0 \geq \mathrm{Re}\,(f-u,\, v-u) + \mathrm{Re}\,(g-v,\, u-v)$$

$$= \mathrm{Re}(f-u,\, v-u) - \mathrm{Re}(g-v,\, v-u)$$

$$0 \geq \mathrm{Re}\,(f-g,\, v-u) + \mathrm{Re}\,(v-u,\, v-u)$$

$$|v-u|_H^{\cancel{2}} \leq \mathrm{Re}\,(f-g,\, u-v) \leq$$

$$\leq |f-g|_H \; \cancel{|u-v|_H}$$

$$|v-u|_H \leq |f-g|_H$$

Cor K a closed vector subspace

$$P_K : H \longrightarrow K$$

$\forall f \in K \qquad \mathrm{Re}\,(f - P_K f, \; v - P_K f) = 0$

$\forall v \in K$

and P_K a bounded linear operator

Pf $\quad u = P_K f$. Then $u \in K$

$$\mathrm{Re}\,(f-u, v-u) \leq 0 \qquad \forall v \in K$$

$$\Updownarrow$$

$$\mathrm{Re}\,(f-u, v) \leq 0 \qquad \forall v \in K$$

$$\mathrm{Re}\,(f-u, -v) \leq 0 \implies \boxed{\mathrm{Re}\,(f-u, v) = 0 \quad \forall v \in K}$$

$u, v \in H$

$\cancel{\mathrm{Re}}(u - P_K u, w) = 0 \qquad \forall w \in K$

$\cancel{\mathrm{Re}}(v - P_K v, w) = 0 \qquad \forall w \in K$

λ, μ

$$(f-u, v) = e^{i\vartheta_0} r_0$$

$$e^{-i\vartheta_0}(f-u, v) = r_0$$

$$(f-u, e^{i\vartheta_0} v) = r_0$$

$$\mathrm{Re}\,(f-u, \underbrace{e^{i\vartheta_0} v}_{v_1}) = 0 = v_0$$

$$(u - P_K u, w) = 0 \quad \forall\, w \in K \qquad \lambda, \mu$$

$$(v - P_K v, w) = 0 \quad \forall\, w \in K$$

$$\Big(\lambda u + \mu v - (\underbrace{\lambda P_K u + \mu P_K v}_{K}), w\Big)_H = 0 \quad \forall\, w \in K$$

$$\Longrightarrow \quad \lambda P_K u + \mu P_K v = P_K (\lambda u + \mu v)$$

Thm $\forall\, f \in H'$ ~~t~~here exists $y \in H$

st $\langle x, f \rangle_{H \times H'} = (x, y)_H$

Pf $T : H \longrightarrow H' \qquad H \ni y \longrightarrow (\cdot, y)_H =: Ty$

$$\big|\langle x, Ty \rangle_{H \times H'}\big| = |(x,y)_H| \leq |x|_H |y|_H$$

$$\Longrightarrow \quad |Ty|_{H'} \leq |y|_H$$

$$\cancel{|y|_H} |Ty|_{H'} \geq \langle y, Ty \rangle_{H \times H'} = (y,y)_H = |y|^{\cancel{2}}_H$$

$$|Ty|_{H'} \geq |y|_H$$

So T is an isometry $H \to H'$

$TH \subseteq H'$ $\qquad$ TH is closed in H'

If $\qquad TH \underset{\neq}{\subset} H'$ $\qquad \exists\, x \in H$, $x \neq 0$

s.t. $\quad \langle Ty, Jx \rangle_{H' \times H''} = 0 \quad \forall\, y \in H$

$$= \underbrace{\langle Ty, x \rangle_{H' \times H}} = 0 \quad \forall\, y \in H$$

$y = x$ $\qquad = (x, y)_H = (x, x)_H = |x|_H^2 > 0$

$$\Rightarrow \quad TH = H'$$

Def $\quad S \subset H$ is orthonormal if

$|x|_H = 1 \quad \forall\, x \in S$ and

$(x, y)_H = 0 \quad \forall\, x \neq y$ in S

Thm $\quad$ Let $S \subset H$ be orthonormal

1) $\quad$ For any $u \in H$

$$\sum_{s \in S} |(u, s)_H|^2 \leq |u|_H^2 \qquad \text{Bessel inequality}$$

2) $V_S = \overline{\mathrm{sp}\{S\}}$. The following are equiv.

a) $u \in V_S$

b) $\sum_{s \in S} |(u,s)_H|^2 = |u|_H^2$ Parseval identity

c) $\sum_{s \in S} (u,s)_H \, s = u$ in H

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3) $\forall u \in H$ the series $\sum_{s \in S} (u,s)_H \, s = P_{V_S} u$

$$\sum_{s \in S} |(u,s)_H|^2 = |P_{V_S} u|_H^2$$ Parseval identity.

Def S and H as before. If $V_S = H$ then S is a basis of H.

Theo Every H admits an orthonormal basis.