Image Processing for Physicists

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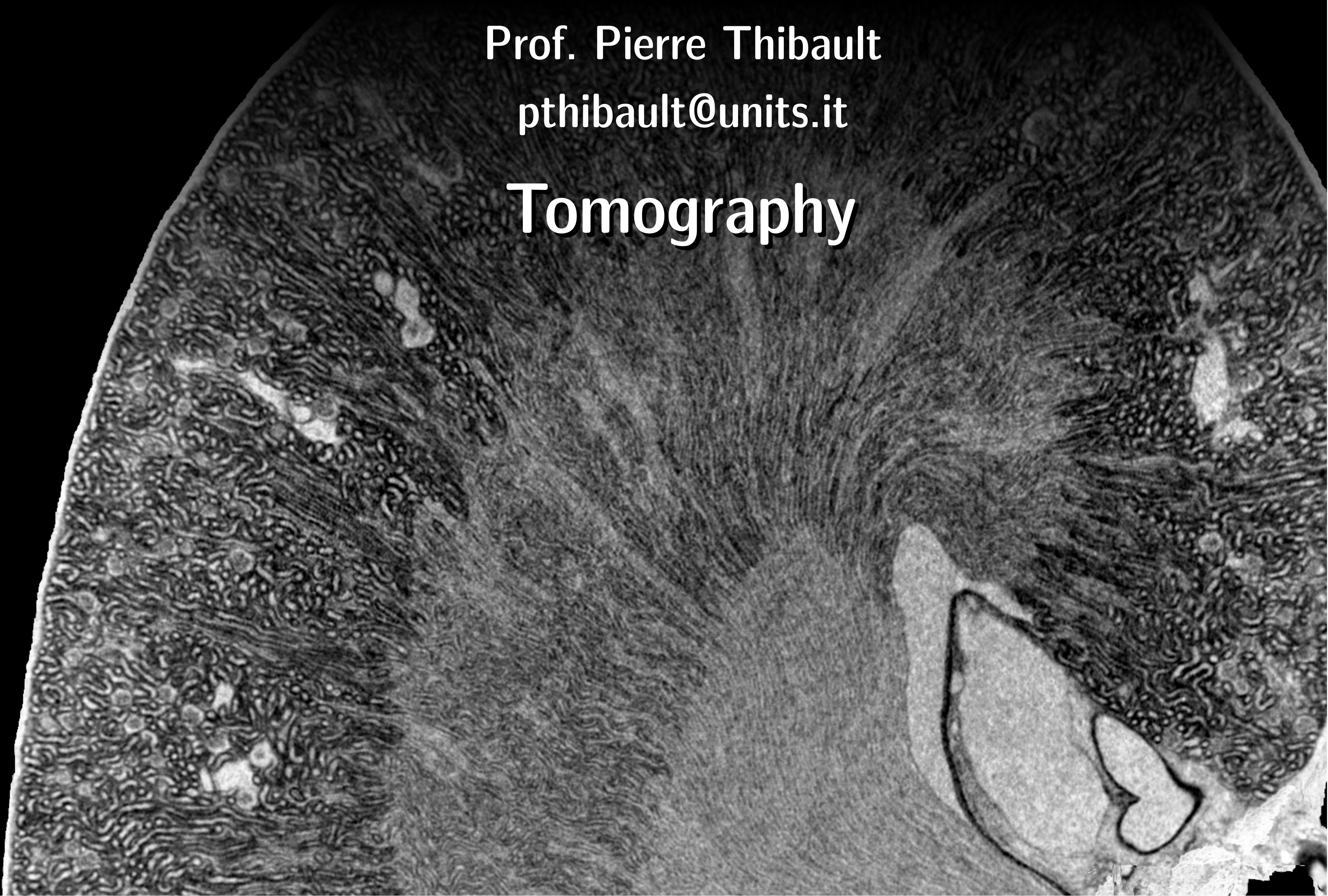
Tomography

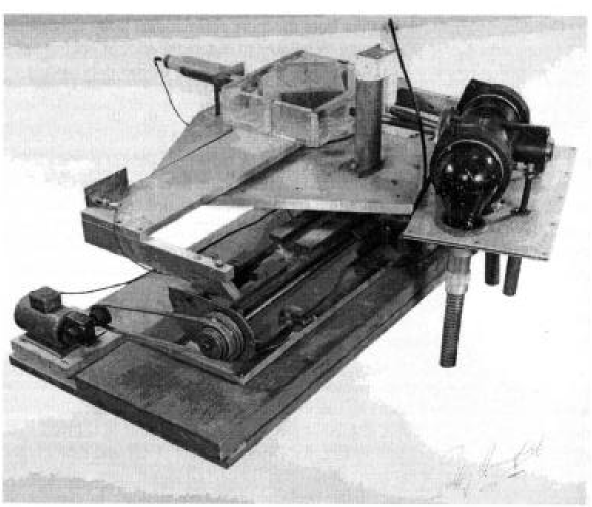


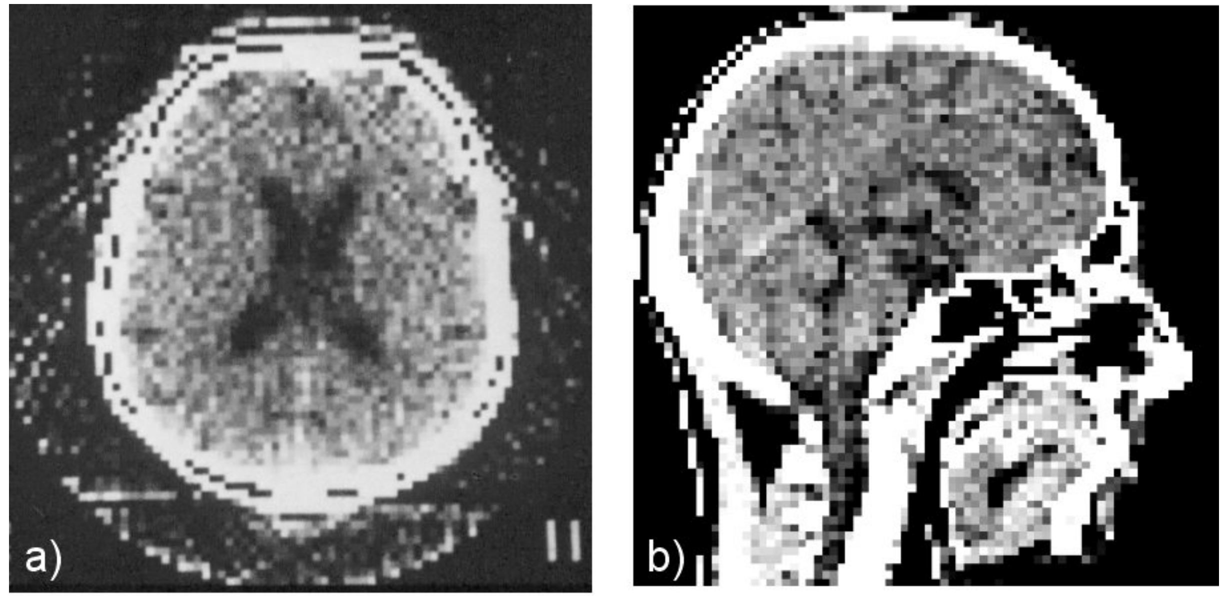
Overview

- Fundamentals of tomography
  - Physics & geometry
- Analytic formulation
  - Radon transform
  - Filtered back-projection
- Algebraic formulation

Examples of tomographic imaging

Computed (X-ray) Tomography (CT)





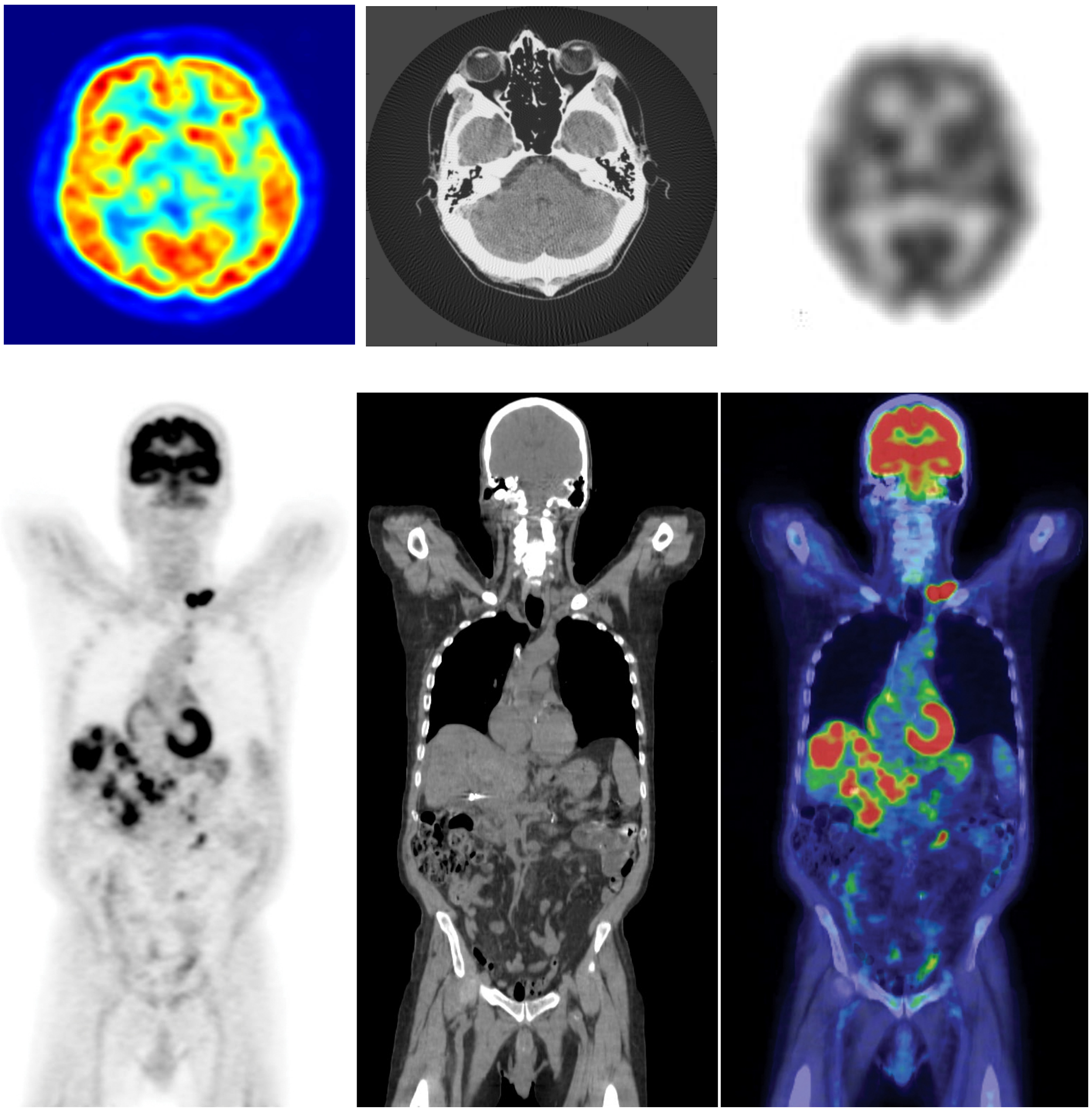
1974, 80x80 pixels

2000, 512x512 pixels, spiral CT

source: W. Kalender, Publicis, 3rd ed. 2011

Examples of tomographic imaging

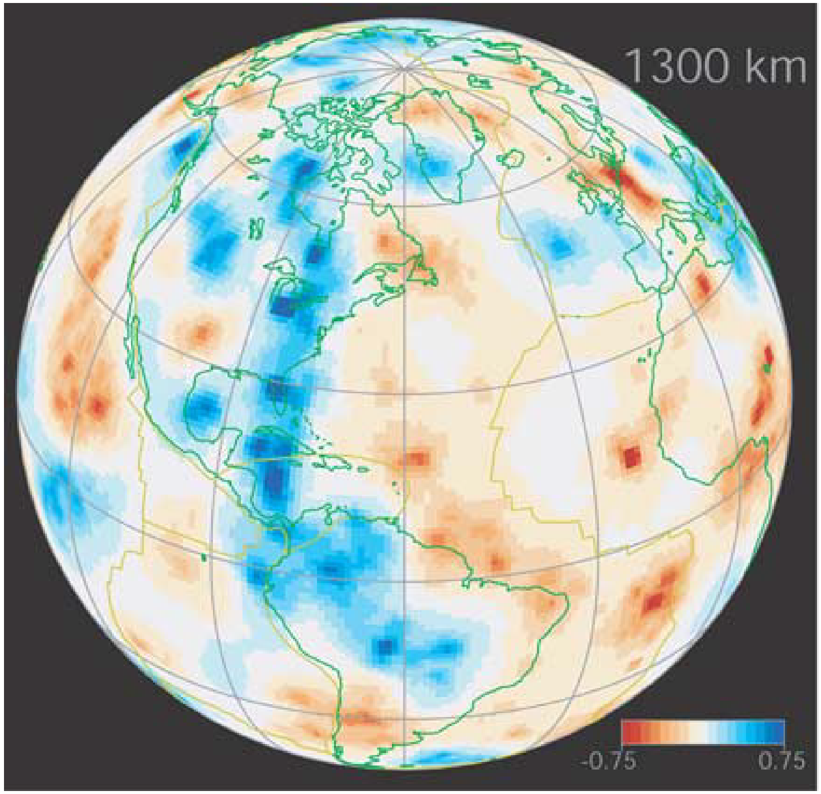
Positron emission tomography (PET) + CT

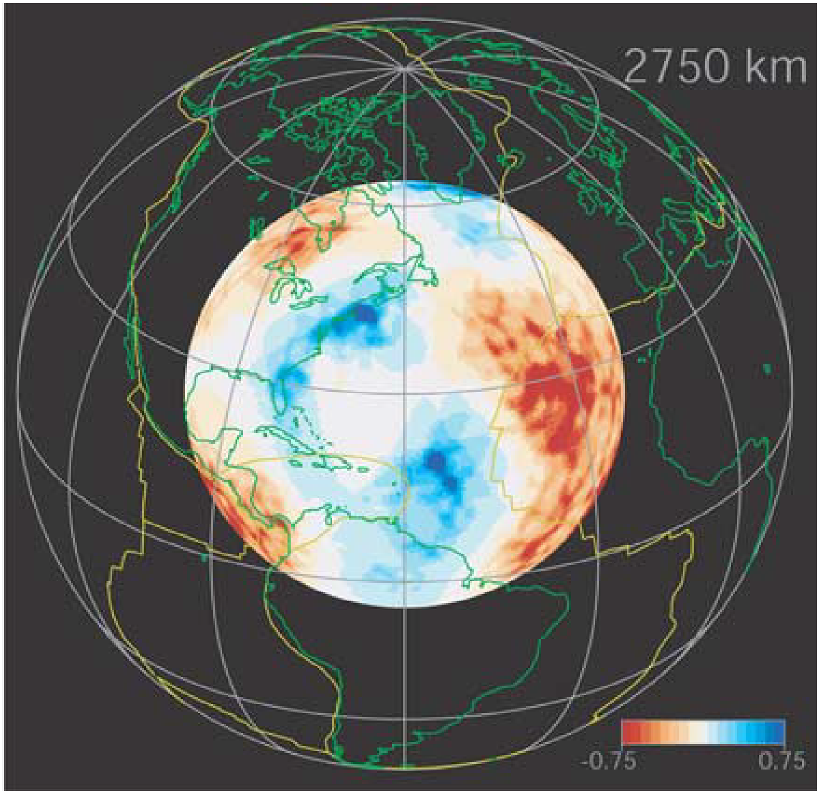


Single-Photon Emission
Computed Tomography (SPECT)

Examples of tomographic imaging

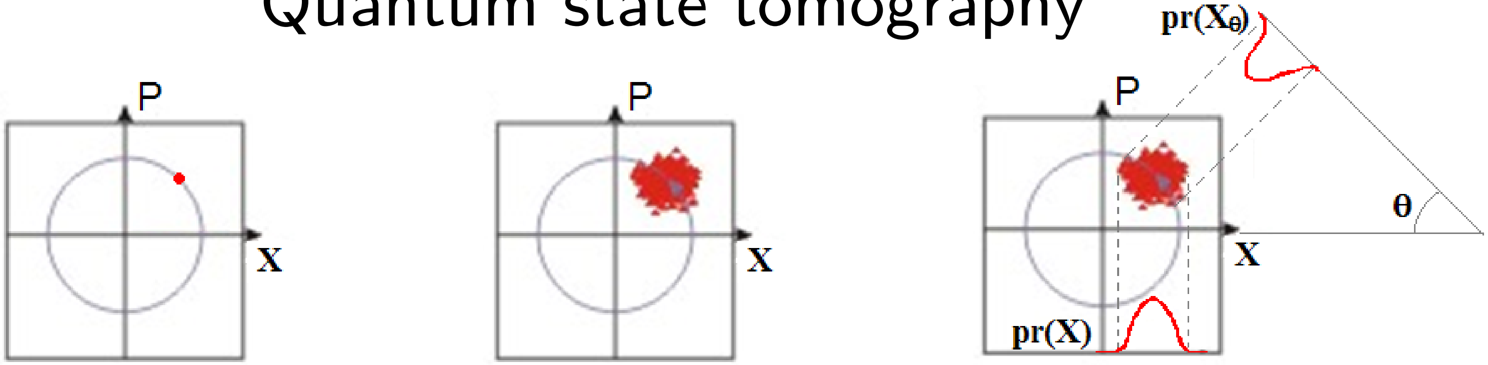
Seismic tomography





source: Sambridge et al. G3 Vol.4 Nr.3 (2003)

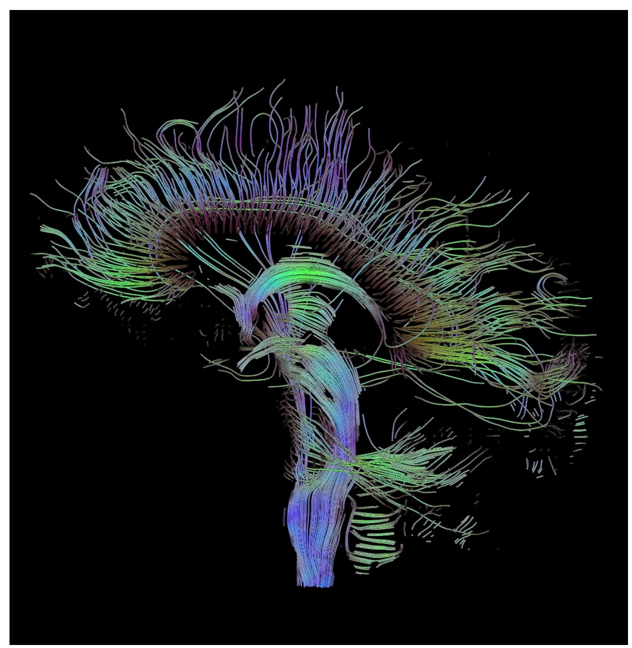
Quantum state tomography



Examples of tomographic imaging

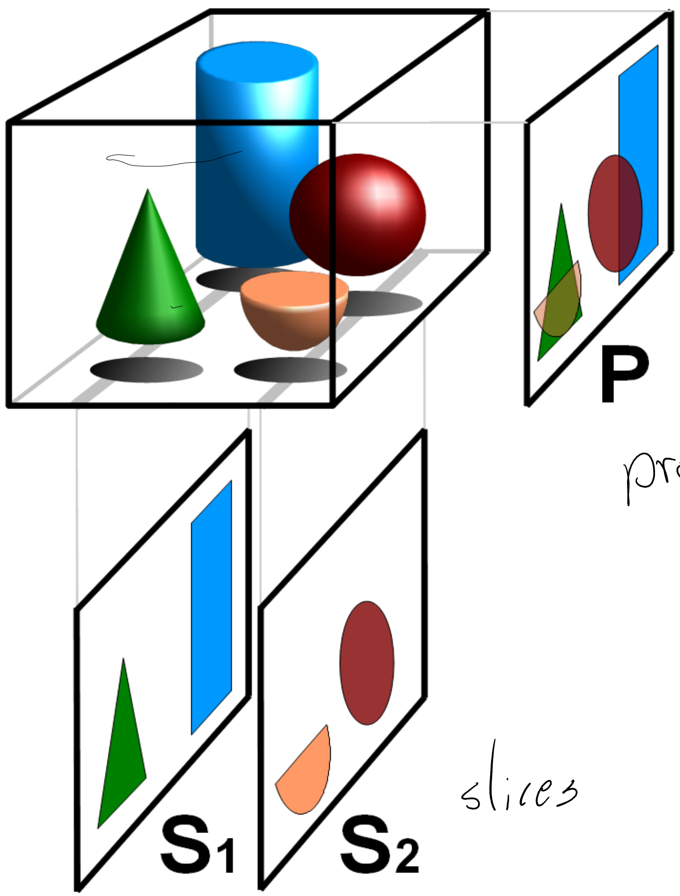
Ultrasonography/tomography (US/UST)

Magnetic resonance imaging/tomography (MRI/MRT)



Reconstructions from projections

Reconstruction of volume from projections

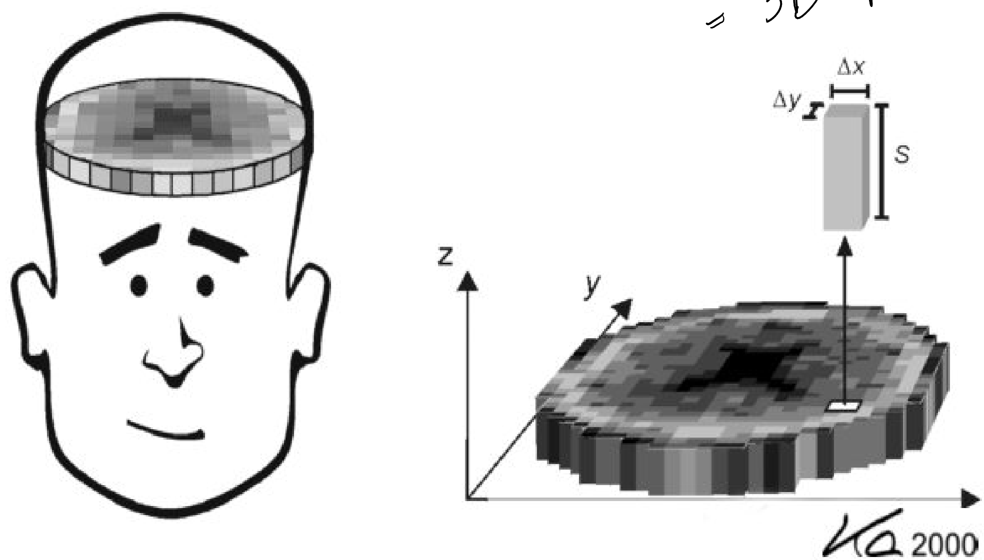


projections

slices

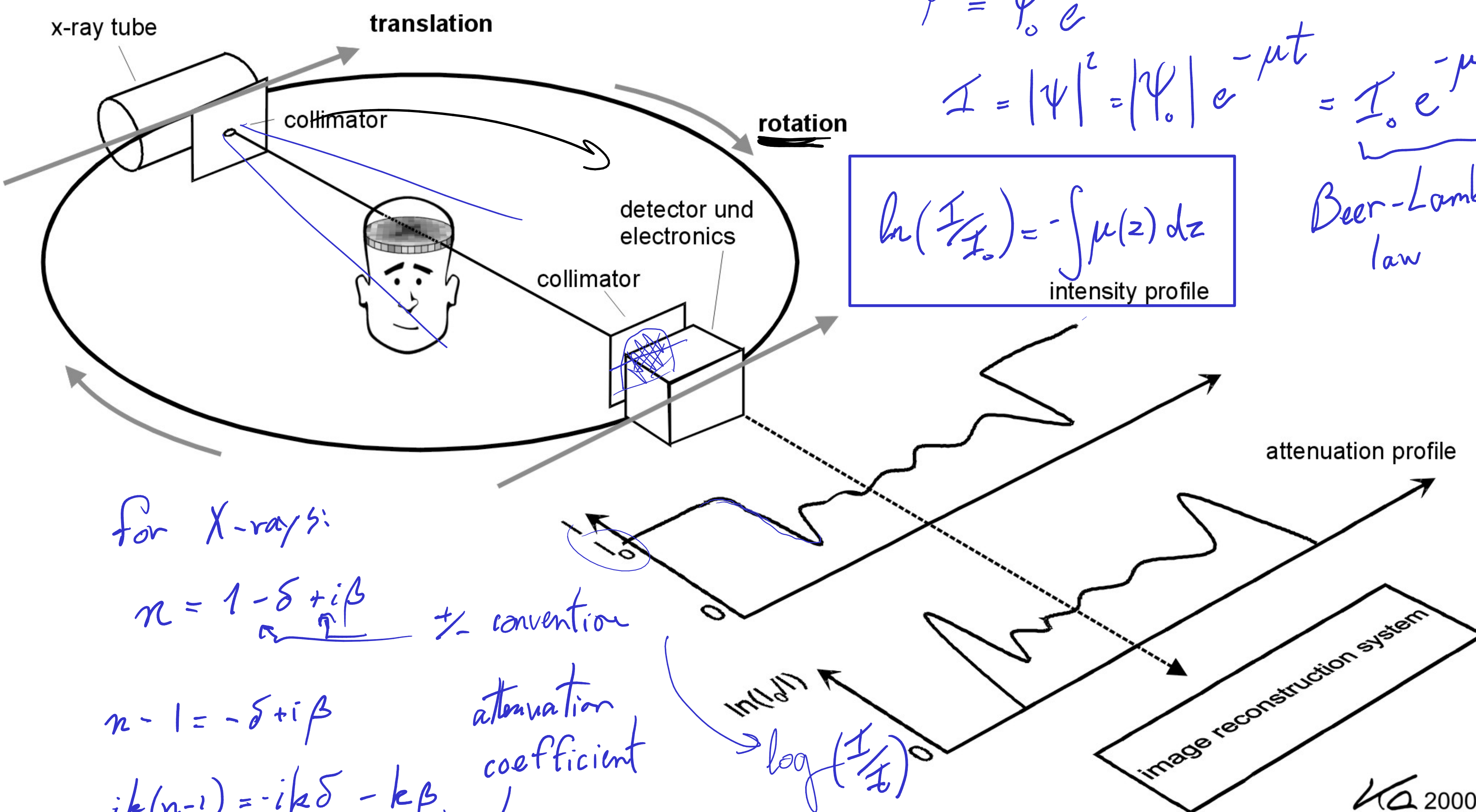
Digitization into voxels

volume elements = 3D pixels



source: W. Kalender, Publicis, 3rd ed. 2011

Principles of X-ray CT



integrated thickness

$$\psi = \psi_0 e^{ik(n-1)t}$$

$$I = |\psi|^2 = |\psi_0| e^{-\mu t} = \underbrace{I_0 e^{-\mu t}}$$

Beer-Lambert law

$$\boxed{\ln\left(\frac{I}{I_0}\right) = -\int \mu(z)\,dz}$$

$$\log\left(\frac{I}{I_0}\right)$$

for X-rays:

$$n = 1 - \delta + i\beta$$

+/- convention

$$n - 1 = -\delta + i\beta$$

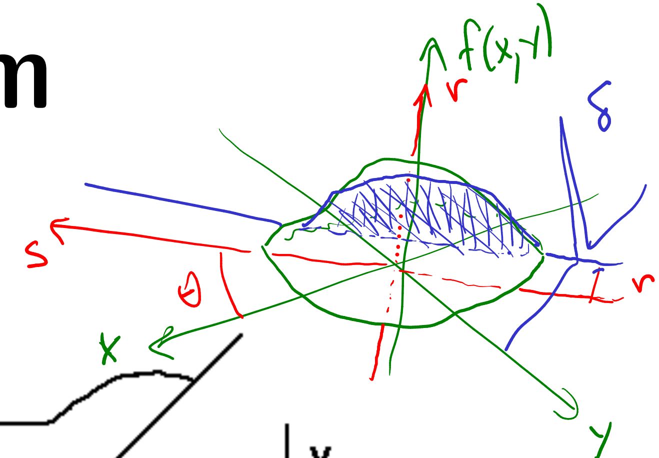
$$ik(n-1) = -ik\delta - \underbrace{k\beta}_{\mu}$$

attenuation coefficient

source: W. Kalender, Publicis, 3rd ed. 2011

In reality μ is spatially dependent: instead of $\mu \cdot t \rightarrow \int \mu(z)\,dz$

Radon transform



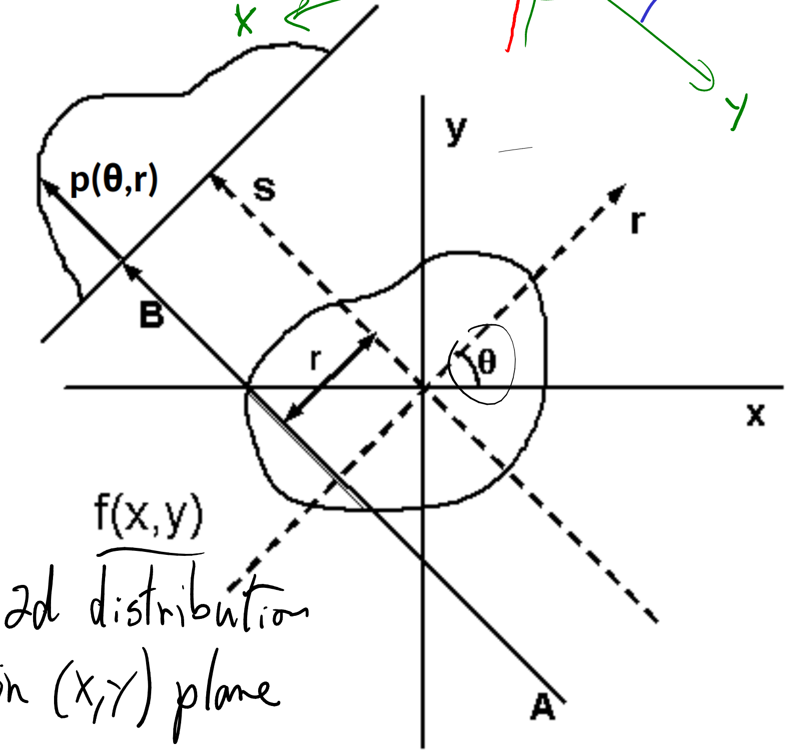
Rotated coordinate system

$$\begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Radon transform

$$p(\theta, r) = \int f(x = r\cos\theta - s\sin\theta,\ y = s\cos\theta + r\sin\theta)\, ds$$

$$= \iint f(x,y)\, \delta(r - (x\cos\theta + y\sin\theta))\, dx\, dy$$



2d distribution in (x,y) plane

* (θ, r) : not the same as polar coordinates (r can be negative!)

line A-B : $\delta(r - (x\cos\theta + y\sin\theta))$

Kak & (Sweeney) ?

$f(x,y) = ?$ given $p(\theta, r)$

inverse Radon transform

$$p(\theta, r) = \int f(x = r\cos\theta - s\sin\theta,\ y = s\cos\theta + r\sin\theta)\, ds$$

$$= \iint f(r'\cos\theta - s\sin\theta,\ s\cos\theta + r'\sin\theta)\, \delta(r - r')\, ds\, dr'$$

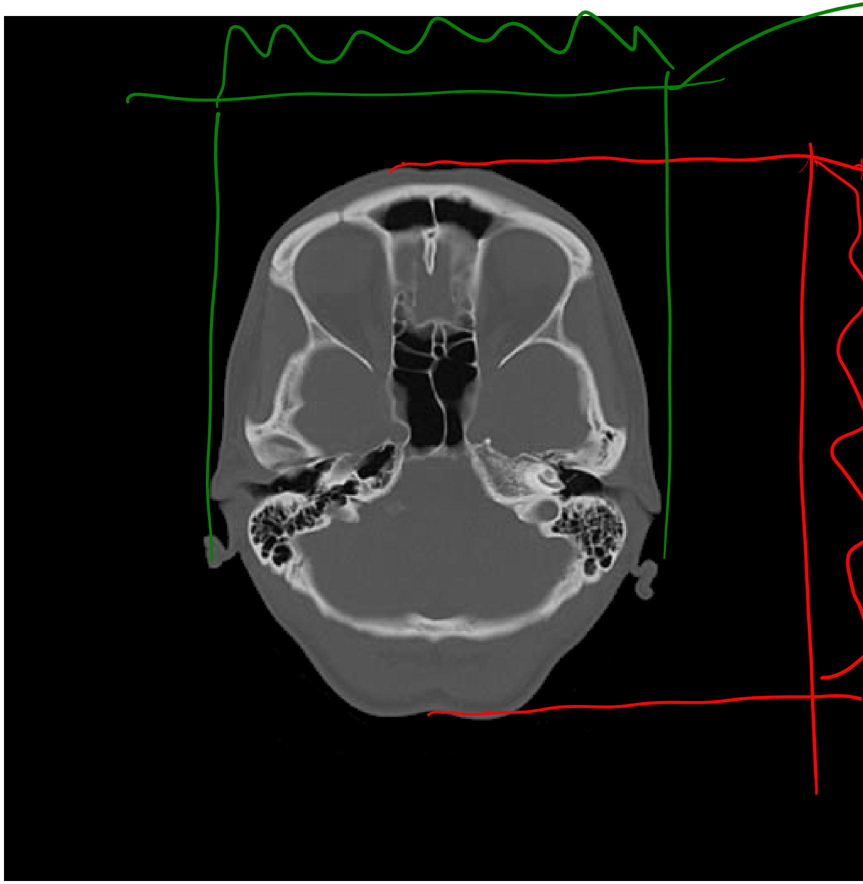
(change of coordinate system $\iint ds\,dr \rightarrow \iint dx\,dy$

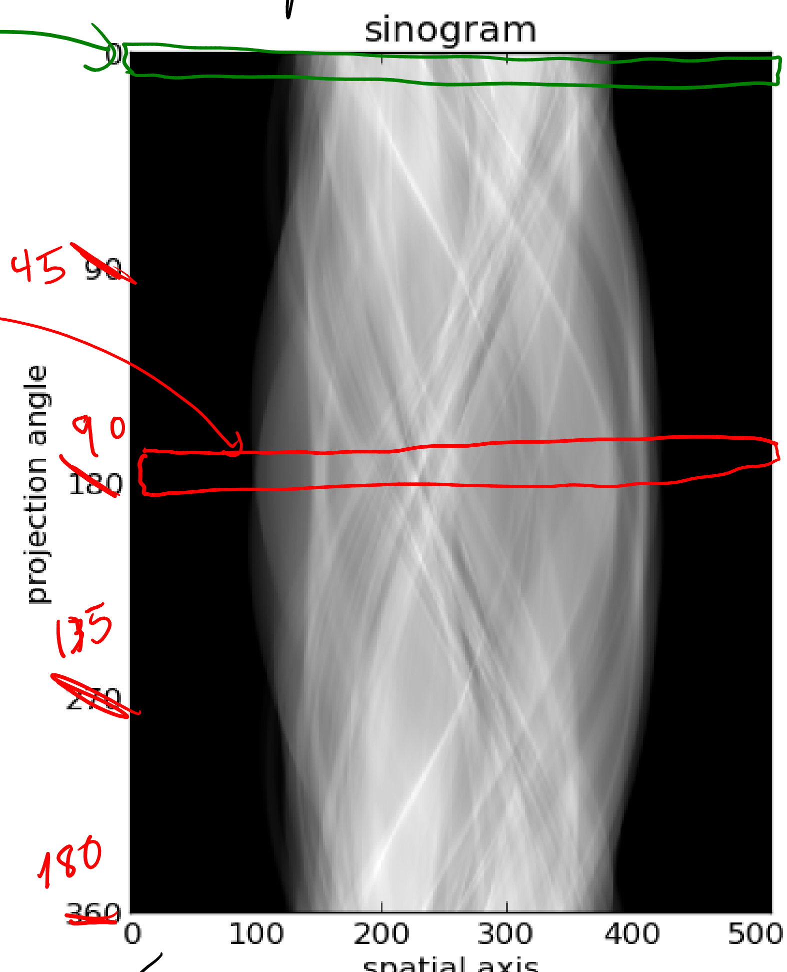
$$= \iint f(x, y)\, \delta(r - (x\cos\theta + y\sin\theta))\, dx\, dy$$

Sinogram

Representation of projection measured by a single detector line as a function of angle

"$f(x,y)$" "$p(\theta,r)$"





θ

r

Computed tomography

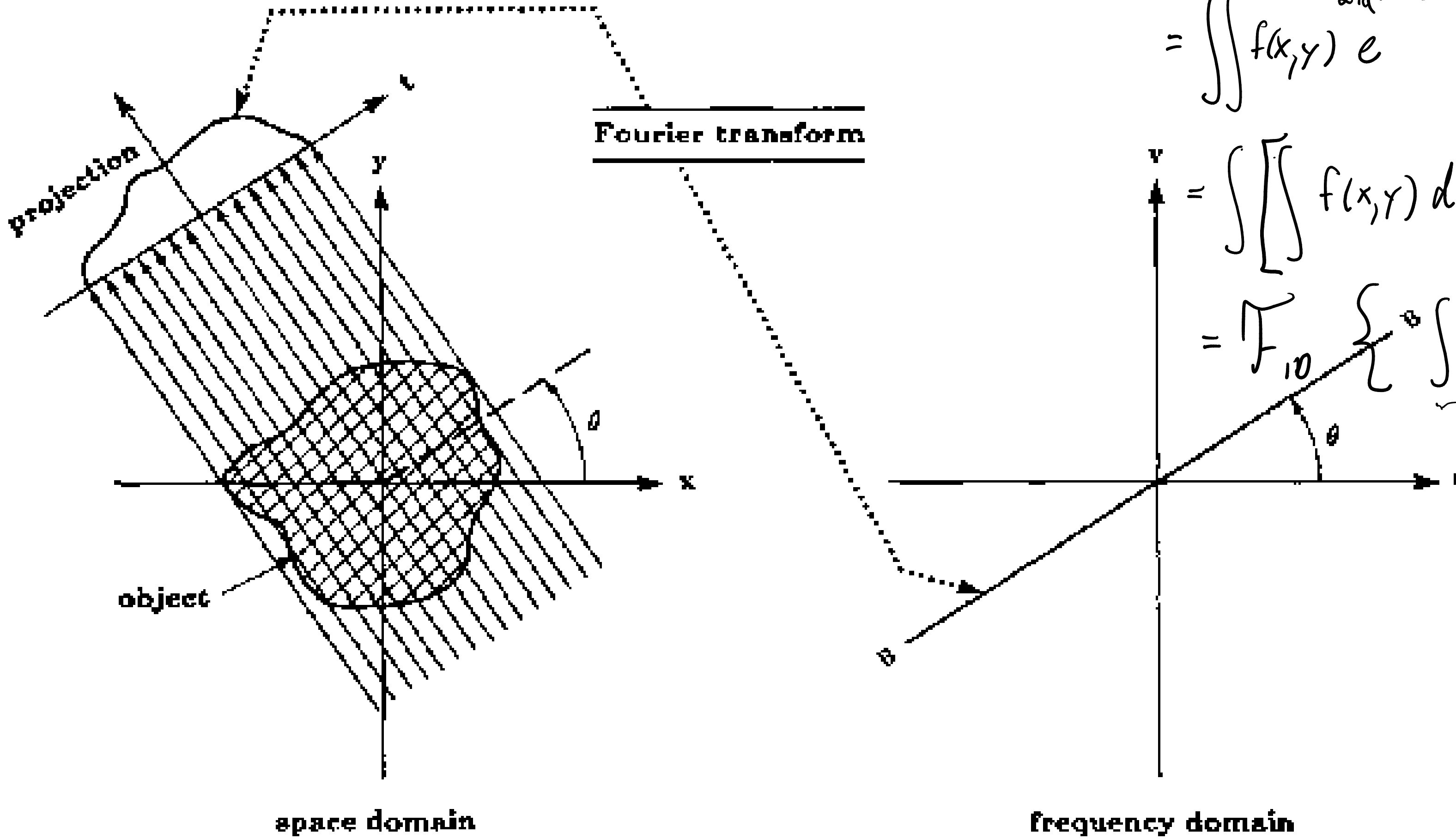
The Fourier slice theorem

e.g. projection along x: $F(u=0, v) = \mathcal{F}_{2D}\{f\}(u=0, v)$

$$= \iint f(x,y)\, e^{2\pi(-i0x - ivy)}\, dx\, dy$$

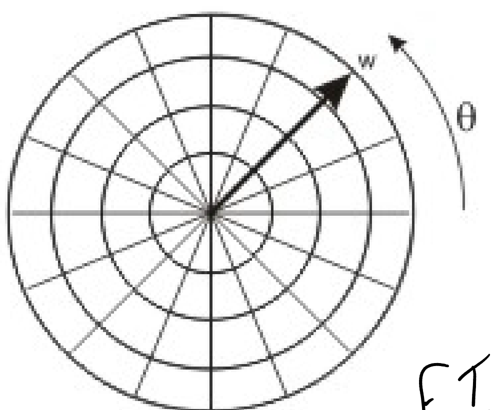
$$= \int \left[\int f(x,y)\, dx\right] e^{-2\pi i v y}\, dy$$

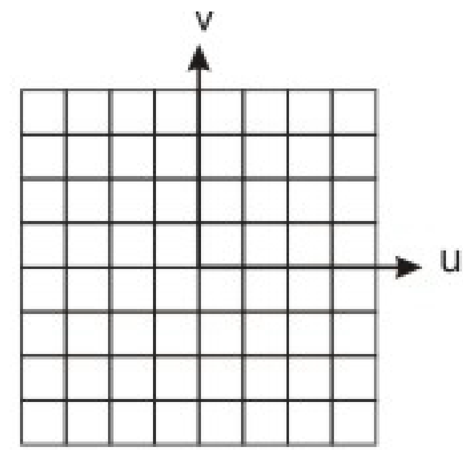
$$= \mathcal{F}_{1D}\left\{\int f(x,y)\, dx\right\}$$



Frequency space sampling

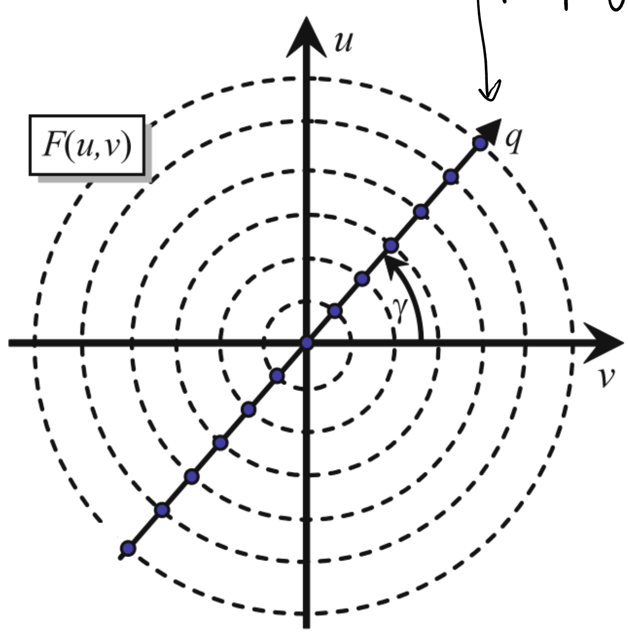
Change of sampling grid from polar to rectangular requires interpolation



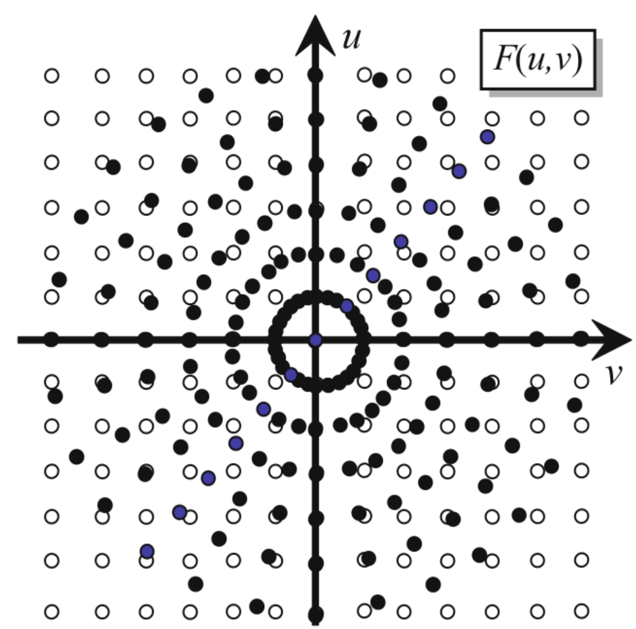


regriding method (really exists and used by some people)

F.T. of one of the projections



a



b

Filtered back-projection

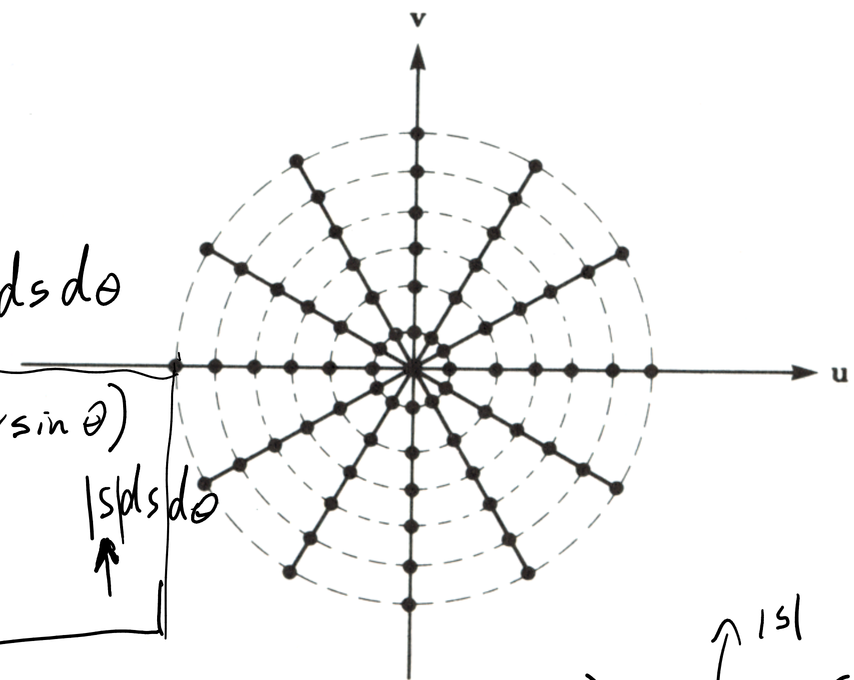
$$f(x,y) = \mathcal{F}^{-1}\left\{ F(u,v) \right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i (ux+vy)}\, du\, dv$$

$$* \quad \int_0^{2\pi}\int_0^{\infty} \longrightarrow \int_0^{\pi}\int_{-\infty}^{\infty}$$

Polar coordinates: $u = s\cos\theta$, $v = s\sin\theta$, $du\,dv = s\,ds\,d\theta$

$$f(x,y) = \int_0^{\pi} \boxed{\int_{-\infty}^{\infty} F(s\cos\theta, s\sin\theta)\, e^{2\pi i s(x\cos\theta + y\sin\theta)}\, |s|\, ds}\, d\theta$$

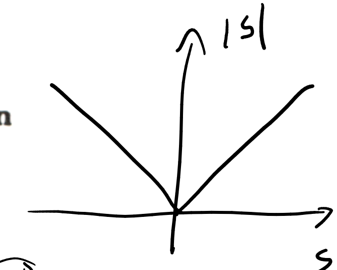


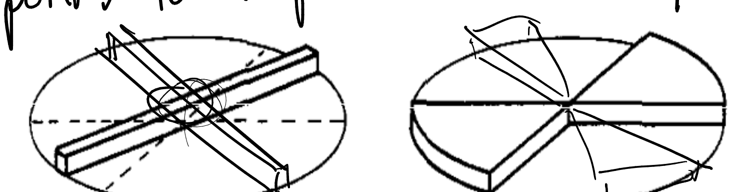
frequency domain

$\theta = 90^\circ$ (for instance)

only difference

$$\int_{-\infty}^{\infty} F(0,s)\, e^{2\pi i s y}\, |s|\, ds \longleftarrow \text{corresponds to ramp filter}$$





$=$ almost $\mathcal{F}^{-1}_{1D}$ to get projection $\int f(x,y)\,dx$

Filtered back-projection

measured (sinogram) $\quad$ 2D F.T. of $F(x,y)$

$p(r,\theta)$ $\qquad$ $F(u,v)$

1D F.T. along r direction

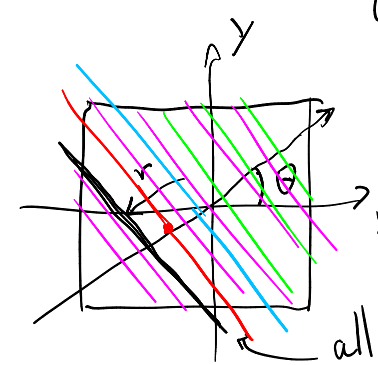
$$\mathcal{F}_{(r)}\{p(r,\theta)\} = F(s\cos\theta, s\sin\theta) \leftarrow \text{Fourier slice theorem}$$

Filtered back-projection:

applying filter onto $p(r,\theta)$

$$\tilde{p}(r,\theta) = \int_{-\infty}^{\infty} F(s\cos\theta, s\sin\theta)\,|s|\,e^{2\pi i r s}\,ds$$

$$\tilde{p}(x\cos\theta + y\sin\theta, \theta) = \int_{-\infty}^{\infty} F(s\cos\theta, s\sin\theta)\,|s|\,e^{2\pi i(x\cdot s\cos\theta + y s\sin\theta)}\,ds$$



all the points with $x\cos\theta + y\sin\theta = r$

$$\Rightarrow \underbrace{\int d\theta\, \tilde{p}(x\cos\theta + y\sin\theta, \theta)} = f(x,y)$$

$$\tilde{p}(\underbrace{x\cos\theta + y\sin\theta}, \theta) \leftarrow \text{function of } (x,y) \text{ for a given } \theta$$

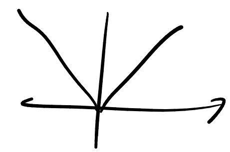
$\Longrightarrow$ spread 1D function in 2D space

Filtered back-projection

Recipe:

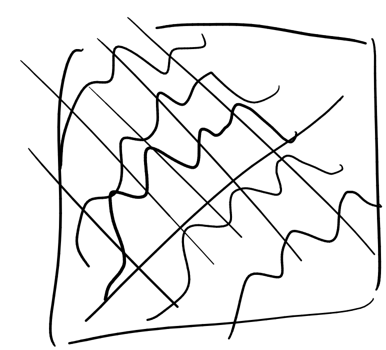
1) FFT of sinogram $p(r,\theta)$ along r (the spatial dimension, not the angle!)

2) multiply with filter $|s|$



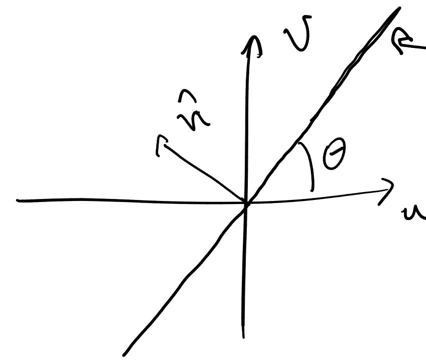
3) Inverse FFT $\rightarrow \tilde{p}(r,\theta)$

4) For each angle θ, back-project $\tilde{p}(r,\theta)$ onto a 2D array and add.



Filtered back-projection

Take a slice of $F(u,v)$ that passes through the origin



line
$u = s\cos\theta$
$v = s\sin\theta$

$$F(s\cos\theta, s\sin\theta)$$

Compute its

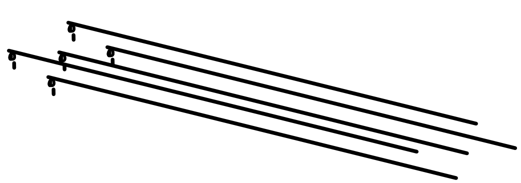
$$\mathcal{F}^{-1}\{F(u,v)\,\delta(u\sin\theta - v\cos\theta)\}$$

$\parallel$

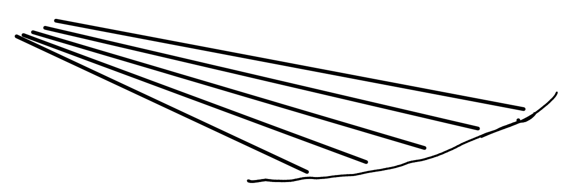
back-projection

$$\iint F(u,v)\,\delta(u\sin\theta - v\cos\theta)\, e^{2\pi i \cdots}\, du\,dv$$

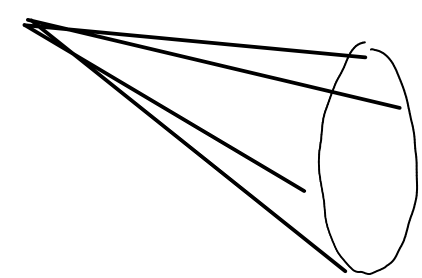
parallel beam



fan beam

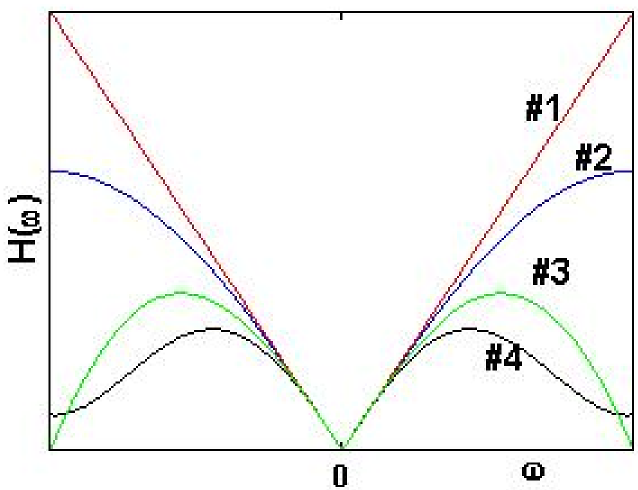


cone beam



Filtered back-projection

- Filter can be tuned to achieve image enhancement
- Trade-off between noise and sharpness

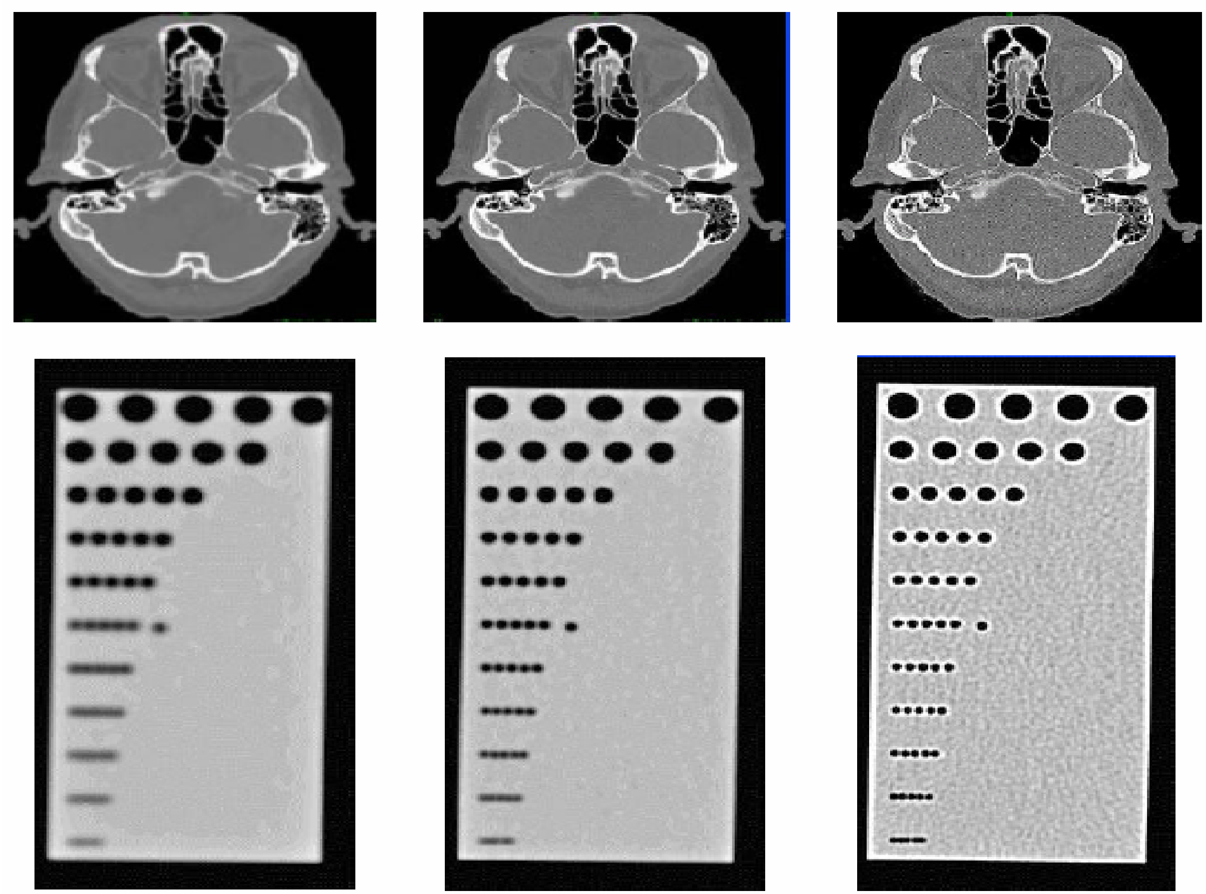


#1 ram-lak (ramp)

#2 Shepp-Logan

#3 cosine

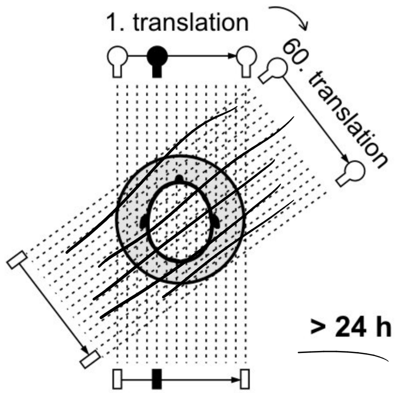
#4 Hamming



smoothing standard edge enhanced

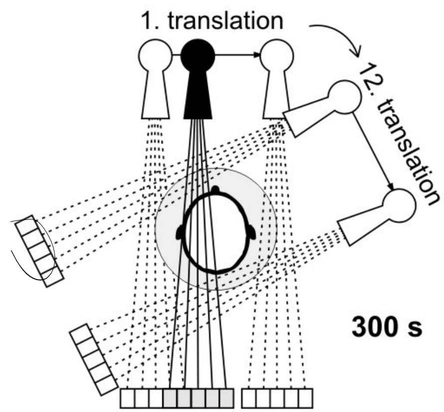
Geometries

pencil beam (1970)

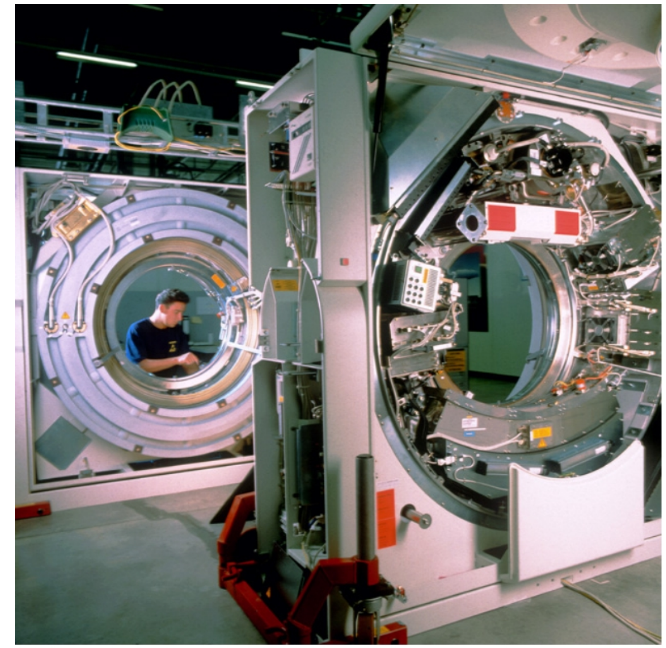


1st generation: translation / rotation

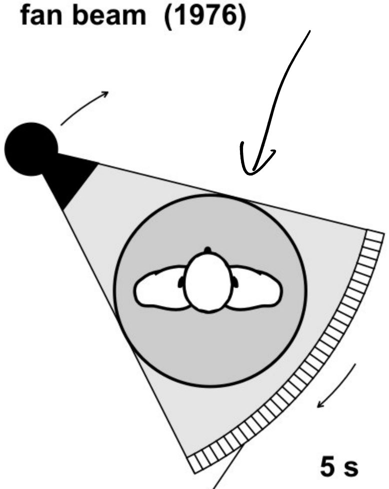
partial fan beam (1972)



2nd generation: translation / rotation

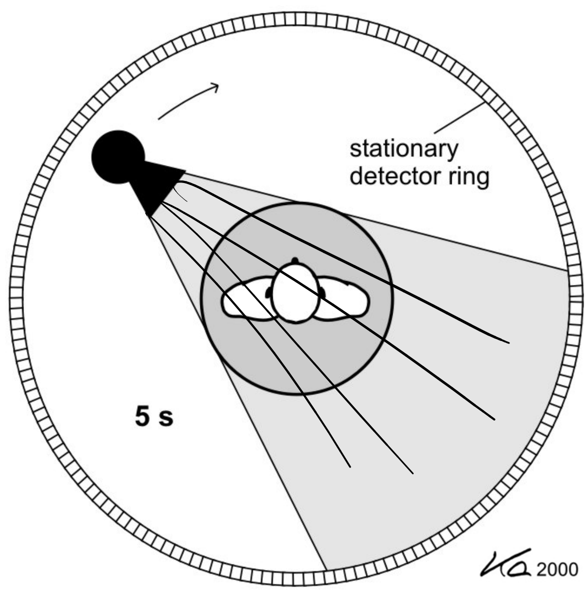


fan beam (1976)

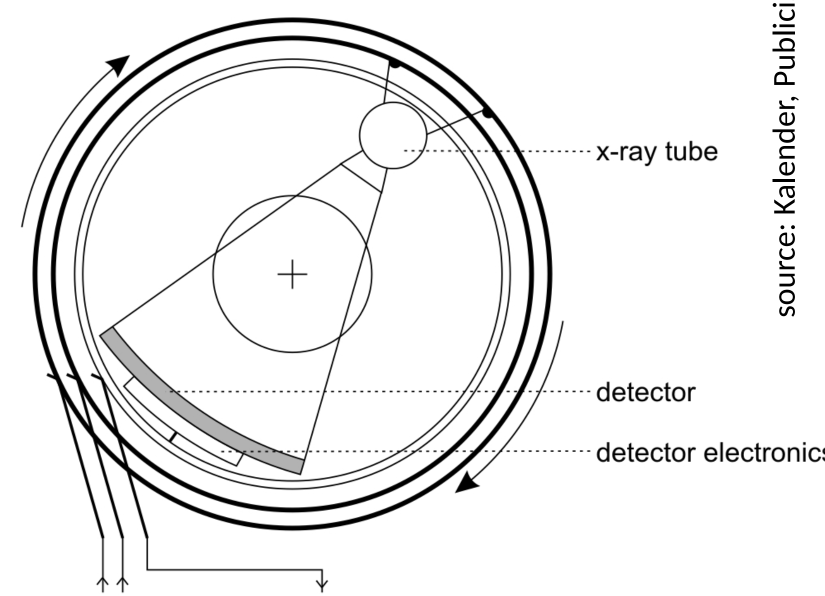


3rd generation: continuous rotation

fan beam (1978)



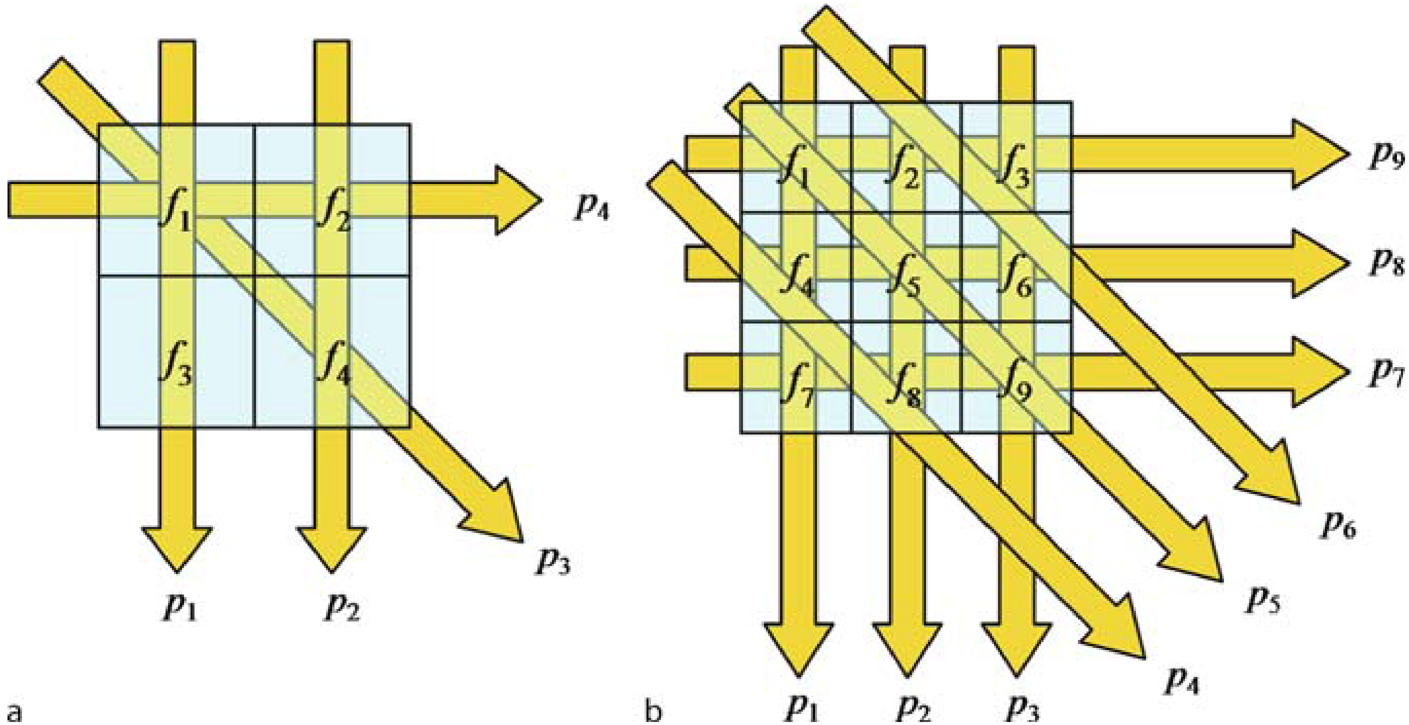
4th generation: continuous rotation



source: Kalender, Publicis, 3rd ed. 2011

Algebraic formulation

Tomography can be formulated as a set of linear equations



$$p_1 = f_1 + f_3$$
$$p_2 = f_2 + f_4$$
$$p_4 = f_1 + f_2$$
$$p_3 = f_1 + f_4$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}$$

"$Ax = b$"

$$x = A^{-1} b$$

source: Buzug, Springer, 1st ed. 2008

Weighting coefficients

Weighting measures:

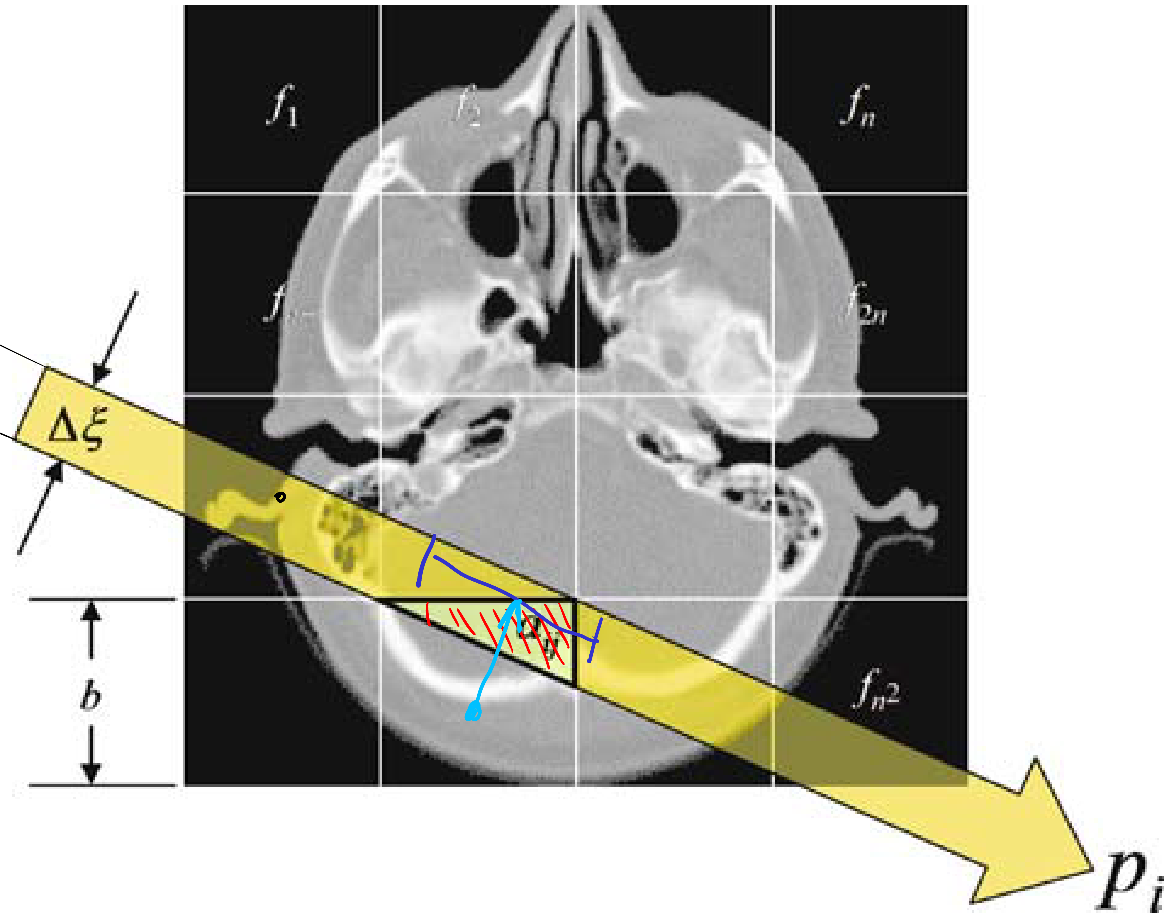
- Logic

  0 or 1

- Area

  overlap area between ray an voxel

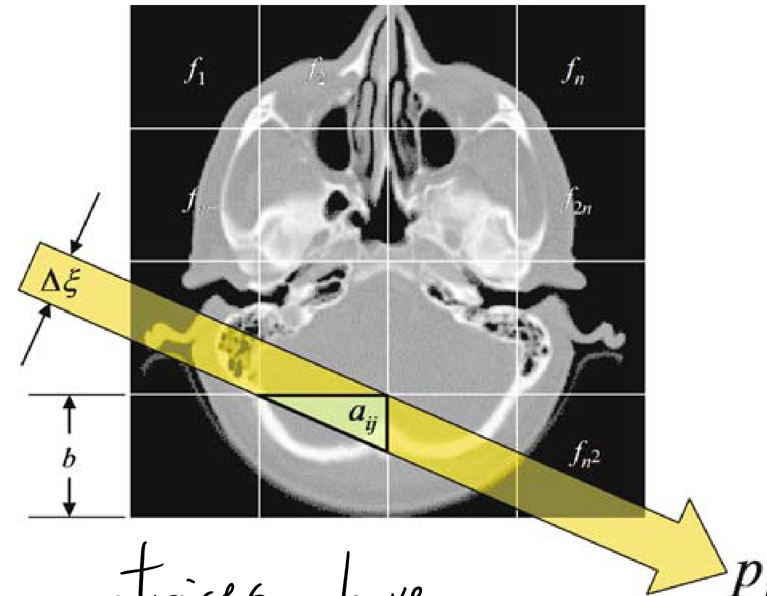
- Path length
- Distance to pixel center



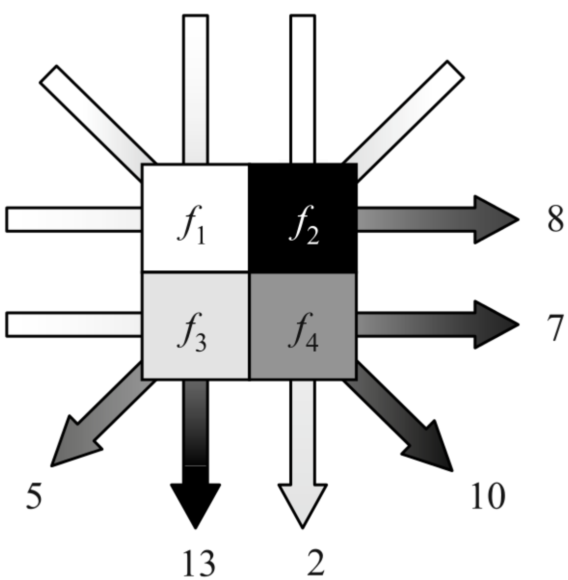
Differences in calculation effort, smoothness, noise sensitivity, ...

source: Buzug, Springer, 1st ed. 2008

System Matrix



actual system matrices have entries between 0 and 1!



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

source: Buzug, Springer, 1st ed. 2008

Matrix (pseudo)-inversion

pseudo inverse

$$M^{-1} = (M^T M)^{-1} M^T$$

Tomographic reconstruction = linear system inversion

$$\begin{array}{ccc} & T & & S \\ \begin{bmatrix} M \end{bmatrix} & \begin{bmatrix} \text{voxels} \end{bmatrix} & = & \begin{bmatrix} \text{sinogram values} \end{bmatrix} \end{array}$$

M is in general <u>not square</u>

$T \sim 1000 \times 1000 = 10^6$

$S \sim 1000 \times 1000 = 10^6$

$M \sim 10^6 \times 10^6$ (10^{12} entries!)

cannot be computed for such large systems

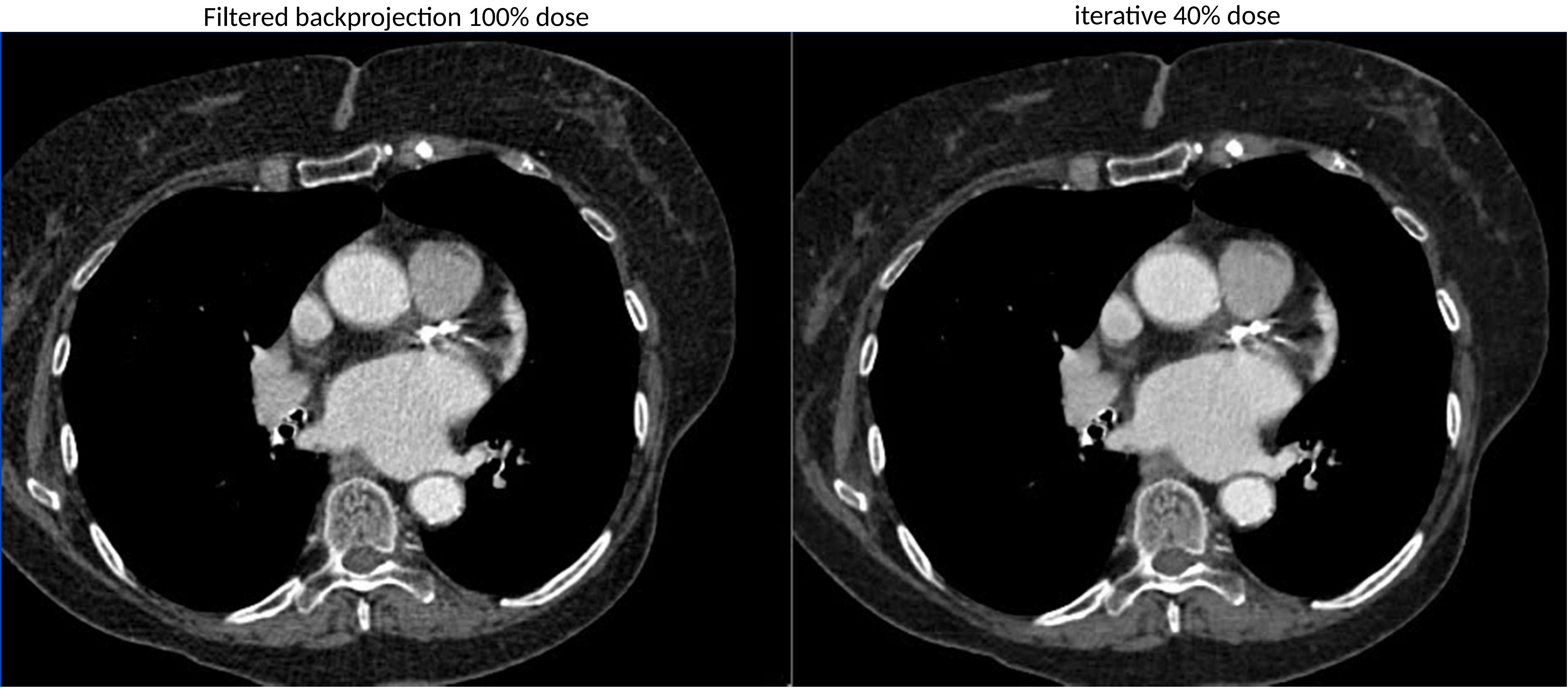
Iterative methods:

- ART Algebraic reconstruction technique
- SART Simultaneous algebraic reconstruction technique
- SIRT Simultaneous iterative reconstruction technique
- MART Multiplicative algebraic reconstruction technique
- MLEM Maximum likelihood expectation maximization
- OSEM Ordered subset expectation maximization
- ... and many, many more

used because often more <u>robust and flexible</u> than FBP.

one can embed more information in the reconstruction process. (constraints)

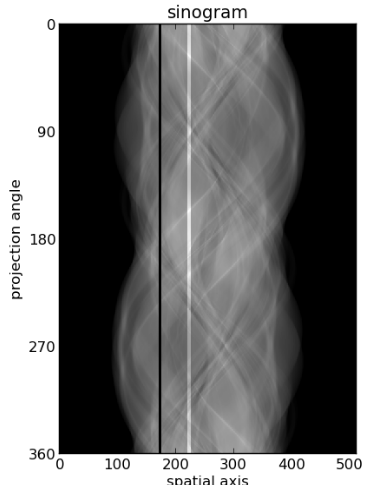
FBP vs algebraic methods

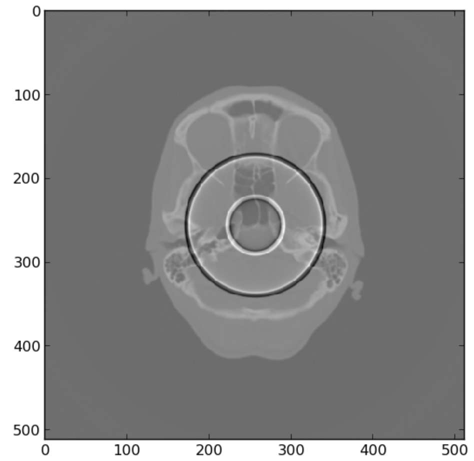


source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

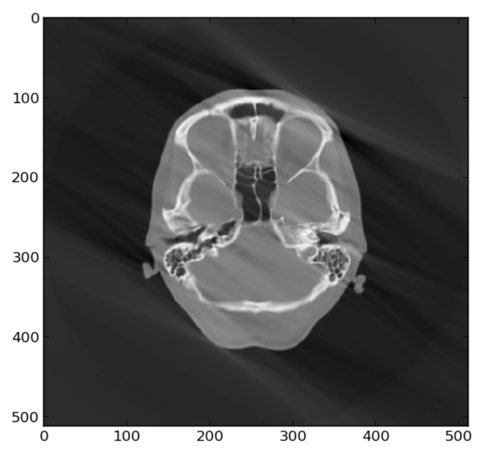
Artifacts

Detector imperfections → ring artifacts



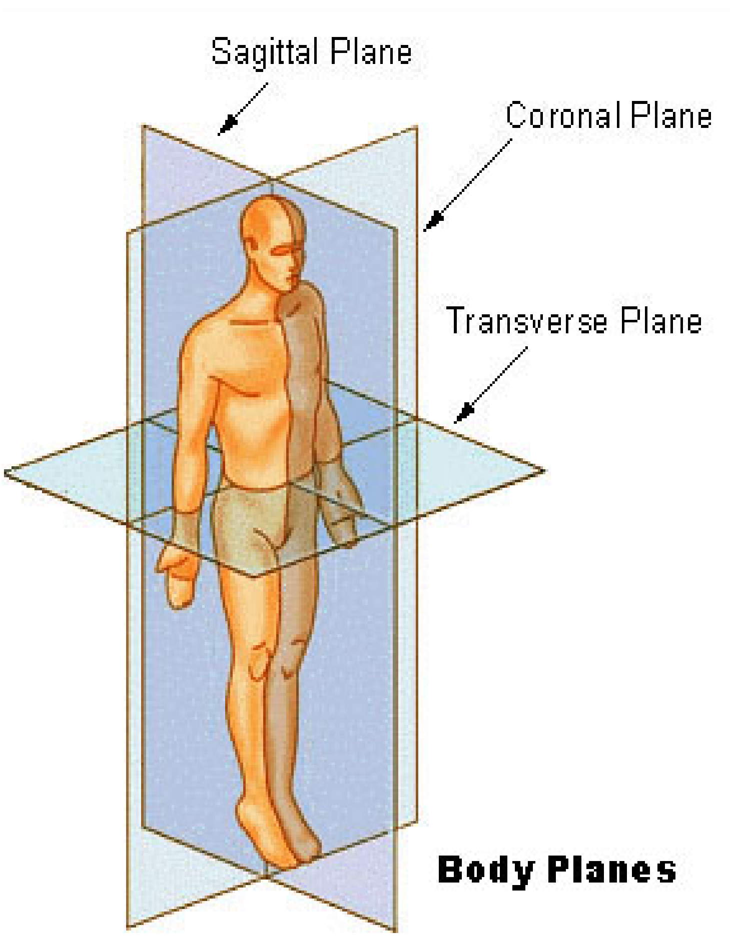


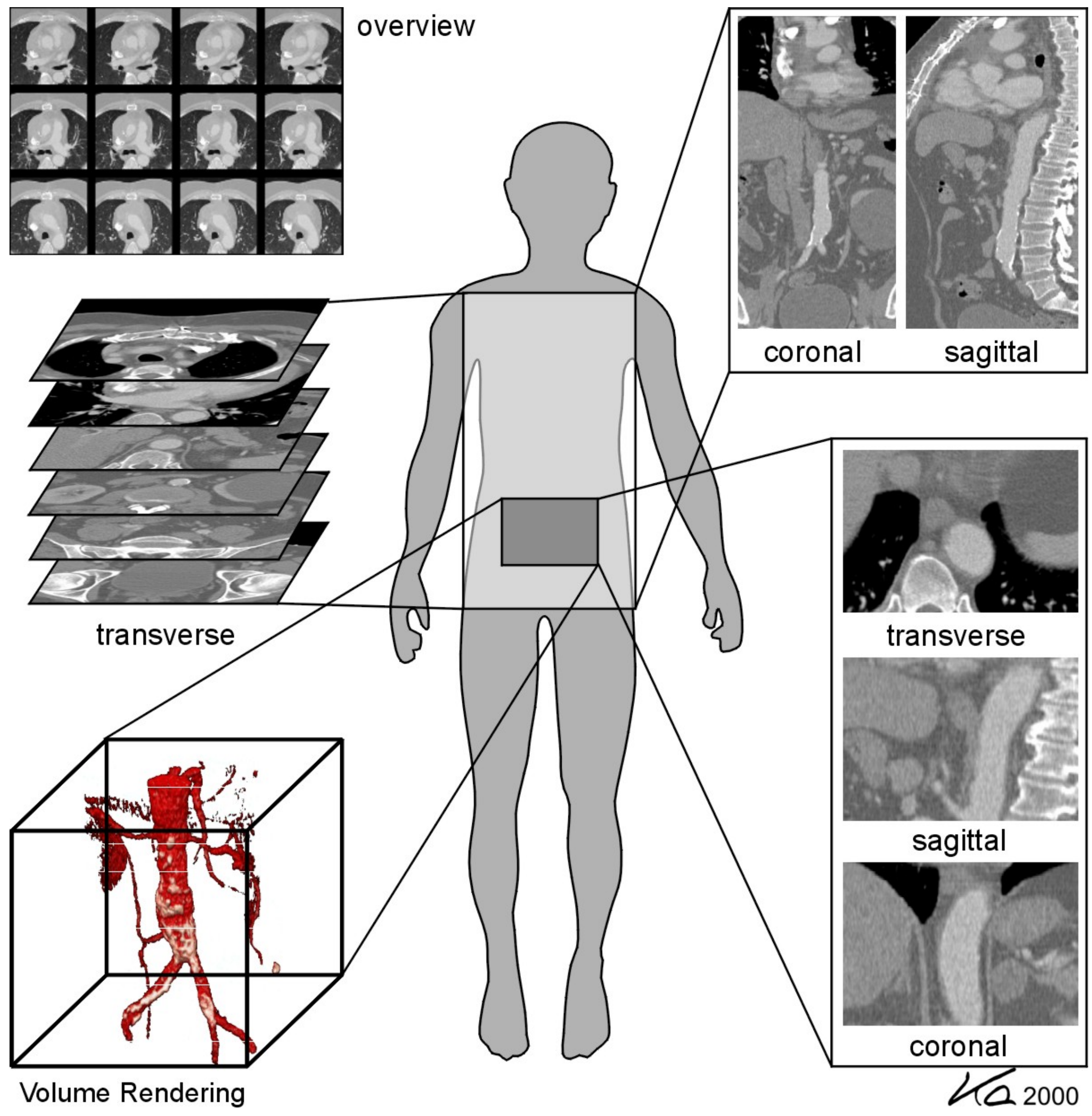
Missing projections → "streak" artifacts



Also: sample motion, beam hardening, ...

Tomographic Display

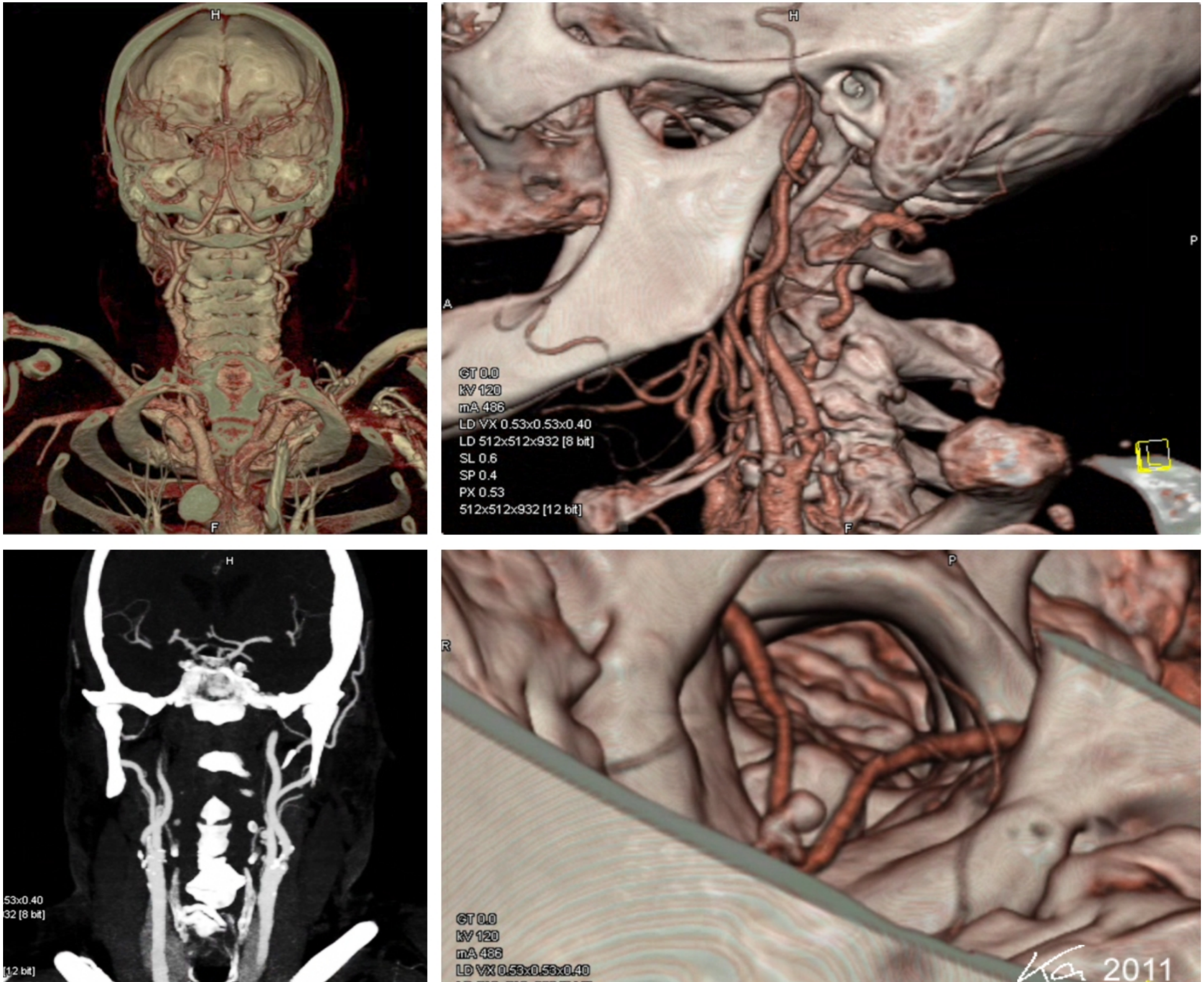




source: http://wikipedia.org

source: W. Kalender, Publicis, 3rd ed. 2011

Volume rendering display



source: Kalender, Publicis, 3rd ed. 2011

Summary

- Computed tomography: reconstruction from projections
- Analytic approach:
  - Projections and tomographic slices are related by the Fourier slice theorem
  - Standard algorithm uses filtered back-projection
- Algebraic approach:
  - Tomography as a system of linear equations
  - Iterative methods are used for large matrix inversions
  - More powerful but computationally more costly
- Imperfect data leads to artifacts