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Introduction

This is a 5-weeks, 10-lectures, swim-or-drown introduction to algebraic geometry, inspired by the book of Kempf, Algebraic geometry: this means that most of the proofs are copied from there, but the material is presented in a somewhat different way. We cover much less material.

If you like algebraic geometry done this way, and want or need to learn it seriously, the standard reference is Hartshorne, Algebraic Geometry, GTM 52. For that, you will need to learn or believe lots of commutative and homological algebra. If you don't like algebraic geometry done this way, you're not alone. Switch to Griffiths–Harris, Principles of algebraic geometry; they'll assume you only know holomorphic functions. Don't be to squeamish if some (or most) of the plus/minus signs are wrong. If you find all those texts too difficult and would like something more elementary with lots of hands-on examples, get a copy of Reid's Undergraduate Algebraic Geometry; be sure to read the last chapter on the history of the subject, maybe comparing it with chapter I.8 of Hartshorne.

The appendix on commutative algebra may or may not be enough for your needs and tastes; when I have time I will include a commented list of reference books.

We will fix an algebraically closed field K throughout. The reader who finds it comforting to assume that $K = \mathbb{C}$ is encouraged to do so.

The notes are in a preliminary version, but they're supposed to be ready soon. Questions, comments and criticism welcome. The parts which are still incomplete are, or will be, labelled POP[1] so that I can more easily find them and fix them.

[1]Prima O Poi.

CHAPTER 1

Algebraic varieties: topology

We start by introducing the objects of our study, namely algebraic varieties. We could do this like for manifolds, as topological spaces together with an atlas of charts. We prefer however to choose a different language, which is also the standard one employed for schemes, namely that of spaces with functions (this is a simplified version of ringed spaces, which we will meet when we define schemes).

In this lecture we will introduce the notion of space with function and start describing the local models of algebraic varieties, namely affine varieties, as topological spaces. In the next lecture we will define regular functions and complete the definition of algebraic variety.

We will fix an algebraically closed field K throughout. The reader who finds it comforting to assume that $K = \mathbb{C}$ is encouraged to do so.

1. Spaces with K-functions

DEFINITION 1.1. A *space with K-functions* is the datum of a topological space X and, for every open subset U of X, of a subset $\mathcal{O}_X(U)$ of the functions from U to K, called *regular functions*. These data is required to satisfy the following conditions:

(1) given an open cover $\{U_i\}_{i\in I}$ of an open subset U, a function $f : U \to K$ is regular if and only if $f|_{U_i}$ is regular for every $i \in I$;
(2) constant functions are regular; if f, g are regular functions on U, then $f+g$ and fg are also regular; $1/f$ is also regular if it is defined (i.e., if $f(x) \neq 0$ for every $x \in U$);
(3) for every open subset U and every regular function f on U, the subset $f^{-1}(0)$ is closed in U.

DEFINITION 1.2. A *morphism of spaces with K-functions* is a continuous map

$$F : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

such that for every open subset V of Y and for every $f \in \mathcal{O}_Y(V)$ one has $f \circ F \in \mathcal{O}_X(F^{-1}(V))$. An *isomorphism of spaces with K-functions* is a morphism with a two-sided inverse.

EXERCISE 1.3. Prove that spaces with K-functions and their morphisms form a category: i.e., show that the identity is a morphism and that composition of morphisms is a morphism.

EXERCISE 1.4. Check that on a one-point set P there is a unique structure of space with K functions given by $\mathcal{O}_P(P) = K$. Prove that any map of a space with K-functions X from and to a point is a morphism.

EXAMPLE 1.5. (1) For any topological field K and any topological space the continuous K-valued functions give X the structure of a space with K-functions.
(2) Let $K = \mathbb{C}$. A C^r (or C^∞, or complex) manifold is a space with K-functions by requesting that C^r (or C^∞) complex valued functions be regular. A morphism of spaces with K-functions among them is a C^r (or C^∞, or holomorphic) map. holomorphic functions is a space with K-functions.
(3) Let X be a space with K-functions and $Y \subset X$ an open subset. Define a structure of space with K-functions on Y as follows: for every open subset U of Y, $\mathcal{O}_Y(U) := \mathcal{O}_X(U)$. Prove that this definition makes Y into a space with K-functions. See also exercise 1.8 below.
(4) An n-dimensional C^∞ manifold is a space with $\mathbb{C}$-functions whose underlying topological space is Hausdorff and paracompact and which is locally isomorphic to the unit open ball in $\mathbb{R}^n$ with C^∞ functions; same for C^r.
(5) An n-dimensional complex manifold is a space with $\mathbb{C}$-functions whose underlying topological space is Hausdorff and paracompact and which is locally isomorphic to the unit open ball of $\mathbb{C}^n$ with the holomorphic functions.

Hence, spaces with K-functions provide an alternative language to the usual one of charts and atlases to define manifolds; it is the preferred one in algebraic geometry since it is the one which extends more easily to cover the case of schemes.

DEFINITION 1.6. Let $(X, \mathcal{O}_X)$ be a space with K-functions, Y a subspace of X. Define a structure of space with K-functions on Y by saying that (for every $U \subset Y$ open) a function $\phi : U \to X$ is regular iff for every $p \in U$ there is an open neighborhood V in X and a regular function $\psi \in \mathcal{O}_X(V)$ such that $\psi|_{V\cap U} = \phi|_{V\cap U}$. I.e., a function is regular on Y if it is locally the restriction of a regular function on X.

EXERCISE 1.7. Prove that in this way $(Y, \mathcal{O}_Y)$ becomes a space with K-functions; we call this the *induced space with K-functions* structure. We also say that Y is a *subspace with K-functions* of X. Prove that for any other space with K-functions Z a set map $F : Z \to Y$ is a morphism iff $i \circ F$ is, where $i : Y \to Z$ is the inclusion. Prove that this space with K-functions structure on Y is the unique one with this property.

EXERCISE 1.8. Prove that if $Y \subset X$ is an open subspace with K-functions, then for every open subset U of Y one has $\mathcal{O}_Y(U) = \mathcal{O}_X(U)$.

DEFINITION 1.9. Let $f : X \to Y$ be a morphism of spaces with K-functions. We say it is a *closed embedding* if it is an isomorphism of X with its image as a closed subspace of Y; analogously we define *open* and *locally closed embeddings*.

EXERCISE 1.10. Prove that an embedded submanifold is a subspace with $\mathbb{C}$ functions.

EXERCISE 1.11. Let X be a space with K-functions, and $\pi : X \to Y$ a surjective map of sets. Prove that there is on Y a unique structure of space with K-functions such that π is a morphism and for any other space with functions Z, and any set map $\psi : Y \to Z$, ψ is a morphism iff $\psi \circ \pi$ is a morphism. Hint: Give Y the quotient topology and decalre a function f to be regular on U iff $f \circ \pi$ is regular on $\pi^{-1}(U)$.

DEFINITION 1.12. Let X be a space with K-functions, and $\pi : X \to Y$ a surjective map. Call Y with the structure of space with functions described in the previous exercise the *quotient spaces with K functions.*

EXERCISE 1.13. Let X be a space with K-functions, $\pi : X \to Y$ a surjective map, Z a subset of Y and $W = \pi^{-1}(Z)$. Prove that the two natural structures on Z (as quoteint of W, itself a subspace of X, and as subspace of Y, itself a quotient of X) agree.

EXERCISE 1.14. Prove that complex projective space (as a complex manifold) has the structure of quotient space with $\mathbb{C}$-functions induced by $\mathbb{C}^{n+1} \setminus \{0\}$.

EXERCISE 1.15. (For future algebraic geometers) Find out what a sheaf is, and prove that regular functions are a sheaf. Prove that the every stalk is a local ring with residue field K.

2. Definition of Zariski topology

Fix a field K. We want to define a topological space $\mathbb{A}^n_K$ (we will also write just $\mathbb{A}^n$ when no confusion can arise).

As a set $\mathbb{A}^n$ is just K^n. For every $p \in \mathbb{A}^n$ and every $f \in K[x_1, \dots, x_n]$ it makes sense to talk about the value of f at p. This suggests a way to go back and forth between subsets of $\mathbb{A}^n$ and of $K[x_1, \dots, x_n]$.

For a subset S of $K[x_1, \dots, x_n]$, its *zero locus* is

$$Z(S) := \{p \in \mathbb{A}^n \mid f(p) = 0 \,\forall f \in S\}.$$

For a subset X of $\mathbb{A}^n$, its *equation set* is

$$E(X) := \{f \in K[x_1, \dots, x_n] \mid f(p) = 0 \,\forall p \in X\}.$$

That is $f \in E(X)$ iff $f|_X = 0$.

LEMMA 1.16. (1) $Z(K[x_1, \dots, x_n]) = \emptyset$;
(2) $Z(\{0\}) = \mathbb{A}^n$;
(3) $Z\left(\bigcup_{j \in J} S_j\right) = \bigcap_{j \in J} Z(S_j)$;
(4) $Z(S_1) \cap Z(S_2) = Z(S_3)$ *where* $S_3 := \{fg | f \in S_1, g \in S_2\}$.

DEFINITION 1.17. The Lemma implies that the subsets of $\mathbb{A}^n$ of the form $Z(S)$ are the closed sets of a topology on $\mathbb{A}^n$, which is called the *Zariski topology*.

REMARK 1.18. We list some easily proven properties of the operations E and Z. Here S and S_i's are always subsets of $K[x_1, \dots, x_n]$ and X and X_i are always subsets of $\mathbb{A}^n$.

- $S_1 \subset S_2 \implies Z(S_2) \subset Z(S_1)$;
- $X_1 \subset X_2 \implies E(X_2) \subset E(X_1)$;

- $X \subset Z(E(X))$;
- $S \subset E(Z(S))$.

Note that for every $X \subset \mathbb{A}^n$, the subset $E(X)$ of $K[x_1, \ldots, x_n]$ is an ideal. In fact, it is a *radical* ideal, that is for every $f \in K[x_1, \ldots, x_n]$ such that $f^m \in E(X)$ for some $m > 0$, then $f \in E(X)$. It is immediate from Remark 1.18 to prove that for a closed subset X of $\mathbb{A}^n$ we have $Z(E(X)) = X$.

3. Nullstellensatz

If K were an arbitrary field, in general $E(Z(I)) \neq I$ even for radical ideals I. However, it is a fundamental result that equality holds (and thus, closed subsets and radical ideals are in natural bijection via E and Z) if K is algebraically closed.

Nullstellensatz is german and means Theorem (about the) zero locus.

THEOREM 1.19 (Hilbert Nullstellensatz – NSS). *Let K be an algebraically closed field. Then for every radical ideal $I \subset K[x_1, \ldots, x_n]$ we have $E(Z(I)) = I$.*

PROOF. Note that the condition that K be algebraically closed is obviously necessary. The proof assumes some facts from algebra, many of which are proven in this notes. Let $I \subset K[x_1, \ldots, x_n]$ be a radical ideal, and $g \in K[x_1, \ldots, x_n]$ but not in I. We want to prove that there exists $p \in Z(I)$ such that $g(p) \neq 0$. Let $A = K[x_1, \ldots, x_n]/I$; recall that to give a K-homomorphism $\phi : A \to K$ means precisely to choose a point $(p_1, \ldots, p_n) \in Z(I)$ (namely, $p_i = \phi(x_i)$): in other words, we want to find a ϕ such that $\phi(g) \neq 0$. Let $A' = A_g = A[y]/yg - 1$. Note that A' is a finitely generated algebra, and it's not zero since g is not nilpotent in A (since I is radical). Use Noether normalization[1] Lemma to find $B \subset A'$ isomorphic to $K[x_1, \ldots, x_r]$ such that A' is a f.g. B-module. Choose any K homomorphism $B \to K$ and lift it by A.18 to a K homomorphism $A' \to B$. Compose with the natural map $A \to A'$ to get $\phi : A \to K$. Since g is invertible in A', its image in K must be nonzero. □

COROLLARY 1.20. *E and Z induce a bijection between closed subsets of $\mathbb{A}^n$ and radical ideals of $K[x_1, \ldots, x_n]$.*

It is easy to see that points in $\mathbb{A}^n$ are closed: in fact, $p = (p_1, \ldots, p_n)$ is the zero locus of the ideal $(x_1 - p_1, \ldots, x_n - p_n)$.

COROLLARY 1.21. *Maximal ideals of $K[x_1, \ldots, x_n]$ are in one-to-one correspondence with points of $\mathbb{A}^n$.*

PROOF. Let m be a maximal ideal. By NSS, since m is proper, its zero locus is nonempty; let $p \in Z(m)$. Then $m \subset E(p)$ and therefore $m = E(p)$ by maximality. Conversely, if p is a point, its ideal is the kernel of the morphism $K[x_1, \ldots, x_n] \to K$ defined by $f \mapsto f(p)$; this morphism is clearly surjective, and its kernel is by definition $E(p)$ which is therefore maximal. □

REMARK 1.22. In fact, points of $\mathbb{A}^n$ are also in one-to-one correspondence with $\mathrm{Hom}_K(K[x_1, \ldots, x_n], K)$.

[1] An algebraically closed field is always infinite. If you don't know why, try to figure it out.

DEFINITION 1.23. An ideal is called *principal* if it is generated by one element; hence it is constituted by the multiples of that element. An open set in $\mathbb{A}^n$ is called *principal* if its complement is the zero locus of a principal ideal.

We call principal also the induced open sets in any subspace of $\mathbb{A}^n$.

EXERCISE 1.24. Show that principal open sets are a basis for the topology of $\mathbb{A}^n$. Show that if an ideal is principal then the polynomial generating it is unique up to scalar multiplication by an element of $K^* = K \setminus 0$.

EXERCISE 1.25. Show that the Zariski topology on $\mathbb{A}^1$ is the finite complement topology. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology $\mathbb{A}^1 \times \mathbb{A}^1$.

CHAPTER 2

Algebraic varieties: regular functions

1. Regular functions on closed subsets of $\mathbb{A}^n$

We want to define on $\mathbb{A}^n$ a structure of space with K-functions. We take the minimal one for which polynomials are regular.

Recall that to every polynomial $f \in K[x_1, \dots, x_n]$ we can associate a function $\mathbb{A}^n \to K$ which we also denote by f (since K is algebraically closed it is infinite, hence the polynomial f can be reconstructed from its induced function; therefore it makes sense to use the same letter for them).

Definition 2.1. Let U be an open subset of $\mathbb{A}^n$. We say that a function $\varphi : U \to K$ is *regular* if for every $p \in U$ there exists polynomials $f, g \in K[x]$ and an open subset V of U disjoint from $Z(g)$ such that $\varphi|_V = (f/g)|_V$. That is, a function is regular if it is locally rational.

Exercise 2.2. Let T be a topological space, $S \subset T$ a subset. Prove that S is closed in T if and only if there is an open cover U_i of T such that $S \cap U_i$ is closed in U_i. Prove the same with open instead of closed.

Lemma 2.3. *If ϕ is regular on U, then $\phi^{-1}(0)$ is closed in U.*

Proof. Being closed is a local property by Exercise 2.2, hence we can assume that there are polynomials $f, g \in K[x_1, \dots, x_n]$ such that $\phi = (f/g)$. Then $\phi^{-1}(0) = Z(f) \cap U$ which is closed by definition. $\square$

Definition 2.4. We define $\mathbb{A}^n$, called *affine n–space*, as a space with K–functions by giving it the Zariski topology and the regular functions defined above. We also consider the induced structure on every locally closed subset of $\mathbb{A}^n$.

The following result is very important; it says that in a significant case, the only global regular functions are the polynomial ones.

Theorem 2.5. *If $X \subset \mathbb{A}^n$ is closed, then every regular function on X is the restriction of a polynomial. In particular $\mathcal{O}_X(X)$ is canonically isomorphic to $K[x_1, \dots, x_n]/E(X)$.*

Proof. Let $A \subset \mathcal{O}_X(X)$ be the algebra of polynomial functions. We want to prove that $A = \mathcal{O}_X(X)$. Let $\varphi \in \mathcal{O}_X(X)$. Let

$$J = \{f \in K[x_1, \dots, x_n] \mid f|_X \cdot \varphi \in A\}.$$

J is clearly an ideal. Moreover, J contains $E(X)$ (since if $f \in E(X)$ then $f|_X = 0$). Let $p \in X$, and $\varphi = f/g$ near p. Then one can find h such that $\varphi = f/g$ on X_h and $p \in X_h$ (because principal open sets are a basis of the topology). Therefore $(gh)|_X\varphi = (fh)|_X$ is in A, and $gh \in J$. Hence $p \notin Z(J)$. As p was arbitrary, $Z(J) \cap X = \emptyset$; since $E(X) \subset J$ we have

$Z(J) \subset X$, hence $Z(J) = \emptyset$ and by weak Nullstellensatz $J = K[x_1, \dots, x_n]$. Therefore $1 \in J$, and $1 \cdot \varphi = \varphi \in A$. □

2. Definition of algebraic variety

DEFINITION 2.6. An *affine (algebraic) variety* is a space with K-functions which is isomorphic to $(X, \mathcal{O}_X)$ for a closed subset X of some affine space. If we want to stress that we have chosen the isomorphism, or equivalently the induced morphism to $\mathbb{A}^n$, we talk about *embedded* affine varieties.

DEFINITION 2.7. An *algebraic variety* is a space with K-functions which has a finite cover by open subspaces which are affine varieties. A *morphism* of algebraic varieties is a morphism of spaces with K-functions.

LEMMA 2.8. *Let X be an algebraic variety, Y a closed subspace. Then Y is an algebraic variety; it is called a* closed subvariety *of X.*

PROOF. The statement is local, so we can reduce to the case where X is affine and in fact a closed subspace of some $\mathbb{A}^n$. The result then follows immediately from Exercise 1.7. □

PROPOSITION 2.9. *Let X be a space with K-functions. To give a morphism from X to $\mathbb{A}^n$ is the same as to give an n-tuple of global regular functions $(f_1, \dots, f_n)$ on X.*

PROOF. Let $F : X \to \mathbb{A}^n$ be a morphism. Then $f_i = x_i \circ F$ is by definition a regular function on X. Conversely, let $f_1, \dots, f_n$ be global regular functions, and define $F : X \to \mathbb{A}^n$ by $F(x) = (f_1(x), \dots, f_n(x))$.
To prove that F is continuous, it is enough to prove that $F^{-1}(Z(g))$ is closed in X for every $g \in K[x_1, \dots, x_n]$. But $F^{-1}(Z(g)) = \phi^{-1}(0)$ where $\phi : X \to K$ is the regular function $g \circ F = g(f_1, \dots, f_n)$ (it is regular because regular functions are a K-algebra, hence every polynomial in regular functions is a regular function). We are done since $\phi^{-1}(0)$ is closed by definition.
To prove that F maps regular functions to regular functions, it is enough (since regularity is a local property) to consider the case of rational functions g/h, $g, h \in K[x_1, \dots, x_n]$ defined where $h \neq 0$; as before, $g \circ F$ and $h \circ F$ are regular, hence so is their ratio where $h \circ F$ is nonzero. □

COROLLARY 2.10. *For every subspace X of $\mathbb{A}^n$ and every space with K-functions Y, a set map $F : Y \to X$ is a morphism of spaces with K-functions iff its components $F_i = x_i \circ F$ are regular functions on Y.*

PROPOSITION 2.11. *Let $\phi : X \to Y$ be a morphism of affine varieties, and $\phi^* : K[Y] \to K[X]$ the induced homomorphism of coordinate rings.*

(1) *ϕ has dense image iff ϕ^* is injective;*
(2) *ϕ is a closed embedding iff ϕ^* is surjective.*

PROOF. To be added, but it's a useful result. □

EXERCISE 2.12. Let $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ be polynomials. Define a map $\phi : \mathbb{A}^n \to \mathbb{A}^{n+m}$ by $f(x_i) = (x_1, \dots, x_n, f_1(x), \dots, f_m(x))$. Prove that the image of f is a closed subset X of $\mathbb{A}^{n+m}$, and that $f : \mathbb{A}^n \to X$ is an isomorphism. Hint: use $\psi(y_1, \dots, y_{n+m}) = (y_1, \dots, y_n)$ to construct an inverse.

EXERCISE 2.13. Prove that translations and linear maps define morphisms. Deduce that every finite dimensional K vector space and affine space has a natural structure of affine algebraic variety.

LEMMA 2.14. *Let X be a closed subvariety of $\mathbb{A}^n$, $f \in K[x_1, \dots, x_n]$. Then the subspace $U = X \cap Z(f)$ is an affine variety, isomorphic to Y, the closed subvariety of $\mathbb{A}^{n+1}$ defined by $Y = Z(E(X), x_{n+1}f - 1)$.*

PROOF. We will give an explicit isomorphism. Define $F : U \to Y$ by $F(x) = (x_1, \dots, x_n, 1/f(x))$. It is immediate to check that for every $x \in U$ $F(x) \in Y$, and the components are clearly regular functions. The inverse of F is $G : Y \to U$ defined by $G(x_1, \dots, x_n, x_{n+1}) = (x_1, \dots, x_n)$. Again, it is immediate to check that this is a morphism, and that F and G are inverse of each other. □

COROLLARY 2.15. *If X is an algebraic variety, every open and every locally closed subspace of X is also an algebraic variety, which we call* open or locally closed *subvariety.*

If X is an algebraic variety which is isomorphic to a locally closed subspace of some affine space, it is called quasiaffine. This concept is less common then affine.

3. Specmax

Let X be an affine variety; we can associate to it its ring of global regular functions, $\mathcal{O}_X(X)$; it is also denoted $K[X]$ and called the *coordinate ring* of X. If $E(X)$ is an equation set for X in $\mathbb{A}^n$, then $K[X]$ is the quotient ring $K[x_1, \dots, x_n]/E(X)$, therefore it's a finitely generated K algebra which is reduced (i.e., it has no nilpotents). Given an affine variety X, the choice of an embedding of X as a closed subvariety of $\mathbb{A}^n$ is equivalent to the choice of n generators for $K[X]$ (the images of $x_1, \dots, x_n$).

LEMMA 2.16. *Let X be an affine variety, Y a ringed space; to a morphism $F : Y \to X$ we associate a homomorphism of K algebras $F^* : K[Y] \to \mathcal{O}_X(X)$. Then $F \mapsto F^*$ induces a bijection between the set of morphisms $Y \to X$ and the set of K-algebra homomorphisms $K[Y] \to \mathcal{O}_X(X)$.*

PROOF. This follows immediately from Corollary 2.10. □

This specializes to

COROLLARY 2.17. *Let X be an affine variety. Then points of X are in bijection with $Hom_K(K[X], K)$.*

We now want to show how to recover an affine variety from its coordinate ring.

Let A be a finitely generated K algebra with no nilpotents. We can construct an affine variety X with A as coordinate ring as follows: the points of X are the maximal ideals of A; the closed subsets are parametrized by ideals I of A, associating to I the locus $Z(I) = \{m \in X \mid I \subset m\}$; a K valued function is regular if it is locally of the form f/g for $f, g \in A$, where $f/g(m) = \phi_m(f)/\phi_m(g)$, and $\phi_m : A \to K$ is the unique morphism of K-algebras having m as kernel.

The ringed space X is called *maximal spectrum* of A, or Specmax(A).

EXERCISE 2.18. Let X be an affine variety, and $A = K[X]$ its coordinate ring. Show that X is canonically isomorphic to $\operatorname{Specmax} A$ via $p \mapsto \ker v_p$ where $v_p : K[X] \to K$ is the homomorphism $v_p(f) = f(p)$.

COROLLARY 2.19. *The category of affine varieties is equivalent to the category of finitely generated K-algebras without nilpotents with arrows reversed.*

CHAPTER 3

Projective and quasiprojective varieties

1. Projective space

As a set projective n-space $\mathbb{P}^n$ is the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ by the $K^*(:= K \setminus \{0\})$ action given by $\lambda(x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$; in other words, it is the set of one dimensional linear subpaces of $\mathbb{A}^{n+1} = K^{n+1}$. One can analogously[1] define $\mathbb{P}(V)$ for any finite-dimensional K-vector space V.

Definition 3.1. Let $\mathbb{A}_0^{n+1}$ be the algebraic variety $\mathbb{A}^n \setminus \{0\}$ and let $\pi : \mathbb{A}_0^{n+1} \to \mathbb{P}^n$ be the structure map. We define *projective n-space* $\mathbb{P}^n$ to be the quotient space with functions.

This means that $U \subset \mathbb{P}^n$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{A}_0^{n+1}$, and that for an open U and a map $f : U \to K$, f is regular iff $f \circ \pi$ is regular on $\pi^{-1}(U)$.

Exercise 3.2. Check that if X is any space with K-function, Y a set, and $\pi : X \to Y$ a map, we can define on Y a unique structure of space with K functions such that for any space with functions Z and any map $g : Y \to Z$, g is a morphism iff $g \circ \pi$ is a morphism. We say that $\pi : X \to Y$ defines Y as a quotient of X. Check that $\pi : \mathbb{A}_0^{n+1} \to \mathbb{P}^n$ mnakes $\mathbb{P}^n$ into a quotient of $\mathbb{A}_0^{n+1}$.

Exercise 3.3. Let $\pi : X \to Y$ be a morphism which makes Y into a quotient of X. Let $V \subset Y$ with the induced subpace structure, and let $U = \pi^{-1}(V)$. Prove that the map $\pi : U \to V$ makes V into a quotient of U.

Theorem 3.4. *The space with functions $\mathbb{P}^n$ is an algebraic variety.*

Proof. We introduce some notation. Let $\tilde{V}_i = \{x \in \mathbb{A}_0^{n+1} \,|\, x_i \neq 0\}$ and $V_i = \{x \in \mathbb{A}_0^{n+1} \,|\, x_i = 1\}$. Note that $\tilde{V}_i$ is affine, with coordinate ring $K[x_0, \dots, x_n][x_i^{-1}]$, and V_i is isomorphic to $\mathbb{A}^n$. Let $U_i = \pi(V_i)$; one has $\tilde{V}_i = \pi^{-1}(U_i)$. Give U_i the structure of space with functions induced by $\mathbb{P}^n$, and note that $\{U_i\}$ is a finite open cover of $\mathbb{P}^n$. To conclude, it is enough to prove that the morphism $\pi_i : V_i \to U_i$ induced by π is an isomorphism of spaces with functions. An inverse to π_i can be constructed explicitly by letting $\alpha_i([v]) = v_i^{-1} \cdot v$. One can check that α_i is well defined and a set-theoretic inverse to π_i; it is a morphism since $\alpha_i \circ \pi_i : \pi^{-1}(U_i) \to V_i$ is a morphism, since its components v_j/v_i are regular functions. □

Definition 3.5. A *projective* (resp. *quasiprojective*) variety is an algebraic variety which is isomorphic to a closed (resp. locally closed) subvariety of some $\mathbb{P}^N$.

[1] Note that in many books $\mathbb{P}(V)$ is the space of hyperplanes in V, i.e. of lines in $V^\vee$.

2. Projective Zariski topology

Let $f \in K[x_0, \dots, x_n]$ be a monomial, $f = a{x_0}_0^i \cdot \dots \cdot x_n^{i_n}$. Its degree is $\deg f = \sum i_j$. A polynomial $f \in K[x_0, \dots, x_n]$ is called *homogeneous* of degree d if every monomial in it has degree d. Homogeneous polynomials form a vector subspace S_d of $S = K[x_0, \dots, x_n]$. In fact, they make S into a graded algebra, since $S_d \cdot S_e \subset S_{d+e}$. Any polynomial f can be decomposed uniquely as sum of homogeneous polynomials of different degrees, called its homogeneous components; i.e., $S = \oplus_{d \geq 0} S_d$.

DEFINITION 3.6. An ideal I is called *homogeneous* if it can be generated by homogeneous elements; equivalently, if for every $f \in I$ each homogeneous component of f is also in I.

DEFINITION 3.7. A *(closed) cone* in $\mathbb{A}^{n+1}$ is a closed subset which contains zero and is invariant under multiplication by scalars. Equivalently, it is the inverse image of a closed subset in $\mathbb{P}^n$ united with zero.

LEMMA 3.8. *There is a natural bijection between closed subsets of $\mathbb{P}^n$ and closed cones in $\mathbb{A}^{n+1}$, given by $Z \mapsto \pi^{-1}(Z) \cup \{0\}$.*

THEOREM 3.9. *Projective NSS. The map $Z \to E(\pi^{-1}(Z))$ induces a bijection between closed subsets of $\mathbb{P}^n$ and radical homogeneous ideals, with the exception of $(x_0, \dots, x_n)$.*

Because of this $(x_0, \dots, x_n)$ is sometimes called the irrelevant ideal.

COROLLARY 3.10. *Let $I \subset K[x_0, \dots, x_n]$ be a homogeneous ideal. Then we can define its zero locus $Z(I)$ as a closed subset in $\mathbb{P}^n$. It is empty if and only if the radical of I contains the irrelevant ideal, or equivalently if there exists $N > 0$ such that $x_i^N \in I$ for every $i = 0, \dots, n$.*

PROPOSITION 3.11. *Let $f, g \in K[x_0, \dots, x_n]$ be nonzero homogeneous polynomials of the same degree d. Then the regular function f/g on $\mathbb{A}^{n+1} \setminus Z(g)$ is the pullback via π of a regular function on $\mathbb{P}^n \setminus Z(g)$, which we also denote by f/g. Conversely, every regular function on an open subset of $\mathbb{P}^n$ can be obtained this way.*

REMARK 3.12. Let $f \in S_d \in K[x_0, \dots, x_n]$ be a homogeneous polynomial of degree $d > 0$. Then $D(f) = \mathbb{P}^n \setminus Z(f)$ is an open subset. As in the affine case, such subsets are a basis of the topology of $\mathbb{P}^n$.

EXERCISE 3.13. Classify endomorphisms of $\mathbb{P}^1$. In particular, show that every automorphism can be written as $f(x_0, x_1) = (a_{00}x_0 + a_{01}x_1, a_{10}x_0 + a_{11}x_1)$ where a is an invertible matrix.

EXERCISE 3.14. Prove that given any three distinct points p_1, p_2, p_3 in $\mathbb{P}^1$ there is a unique automorphism ϕ of $\mathbb{P}^1$ such that $\phi(p_1) = 0 := (1, 0)$, $\phi(p_2) = 1 := (1, 1)$, $\phi(p_3) = \infty := (0, 1)$.

PROPOSITION 3.15. *Let $f_0, \dots, f_r \in K[x_0, \dots, x_n]$ be homogeneous polynomials of degree d, and let $U \subset \mathbb{P}_n$ the open subset whose complement is $Z(f_0, \dots, f_r)$. Then the morphism $f : \pi^{-1}(U) \to \mathbb{A}_0^{r+1}$ defined by $(f_0, \dots, f_r)$ induces a morphism $U \to \mathbb{P}^r$, which we also denote by f.*

3. Veronese embedding

Fix $n > 0$ and $d > 0$, and let $N = \begin{pmatrix} n+d \\ d \end{pmatrix} - 1$. Choose a basis $f_0, \ldots, f_N$ of the vector space $S_d \subset K[x_0, \ldots, x_n]$ of degree d homogeneous polynomials (for instance, you could take all monic monomials in a suitable order, eg lexicographic). Note that $Z(f_0, \ldots, f_N) = \emptyset$, since among the f_i's are all monomials x_i^d.

Definition 3.16. Let $f : \mathbb{P}^n \to \mathbb{P}^N$ be the morphism defined by $(f_0, \ldots, f_N)$. It is called *Veronese embedding* of degree d.

We index the coordinates in $\mathbb{P}^N$ by multiindices $I = (i_0, \ldots, i_n)$ such that $\sum i_k = d$. In particular we write I_k for the multiindex with kth component d and all others equal to zero.

Lemma 3.17. *Let X be the image of the Veronese morphism. Then $\bar{X}$ is contained in*

$$\bigcup_{k=0}^{n} \{y_{I_k} \neq 0\}.$$

Proof. Let $Y \subset \mathbb{P}^N$ be the zero locus of the polynomials $\left(y_I^d - \prod y_{I_k}^{i_k}\right)$, for all multiindices I. Clearly Y contains X, so it contains $\bar{X}$. It is enough to prove that for every $y \in Y$ there exists k with $y_{I_k} \neq 0$. We know that there exists I such that $y_I \neq 0$. Choose k such that $i_k \neq 0$; then y_{I_k} must also be nonzero, since it is a factor of y_I^d. □

Theorem 3.18. *Let $f : \mathbb{P}^n \to X$ be the Veronese morphism. It is an isomorphism, and the image is closed in $\mathbb{P}^N$.*

Proof. Let $U_k = \{y \in \mathbb{P}^N \,|\, y_{I_K} \neq 0\}$; let $X_k = X \cap U_k$ and $V_k = f^{-1}(X_k)$. By the lemma it is enough to prove that X_k is closed in U_k and that $f : V_k \to X_k$ is an isomorphism. By symmetry we can assume $k = 0$. $V_0 = \{x \in \mathbb{P}^n \,|\, x_0^d \neq 0\} = \{x \in \mathbb{P}^n \,|\, x_0 \neq 0\}$. Hence V_0 is isomorphic to $\mathbb{A}^n$ with coordinates $t_i = x_i/x_0$, U_0 is isomorphic to $\mathbb{A}^N$ and the morphism $f : V_0 \to U_0$ is given by $(t_1, \ldots, t_n, \ldots)$. Here t_i appears as ratio $(x_0^{d-1}x_i/x_0^d)$. We are done by [give a reference here!]. □

Corollary 3.19. *Let $f \in S_d \subset K[x_0, \ldots, x_n]$ be a nonzero homogeneous polynomial of degree d. Let $X \subset \mathbb{P}^n$ be a closed subset. Then $X \setminus Z(f)$ is an affine variety, and every regular function on it can be written as g/f^r for some $r > 0$ and some homogeneous polynomial g of degree dr.*

Definition 3.20. The image of the d-th Veronese embedding of $\mathbb{P}^1$ is called rational normal curve of degree d in $\mathbb{P}^d$.

Exercise 3.21. Let C_d be the rational normal curve of degree d. Show that one can write explicit homogeneous equations for C_d by

$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 & \cdots & y_{d-1} \\ y_1 & y_2 & \cdots & y_d \end{pmatrix} \leq 1.$$

Show that they generate the ideal of C_d. Deduce that the ideal of C_d can be generated by polynomials of degree 2 (i.e, C_d is cut out by quadrics).

EXERCISE 3.22. Consider the degree 2 Veronese embedding of $\mathbb{P}^n$. Prove that the image can be seen as the projectivization of the locus of rank 1 symmetric $(n+1) \times (n+1)$ matrices, and that the Veronese map can be written as $x \mapsto x \cdot x^t$. Deduce that also in this case the image of the Veronese embedding is cut out by quadrics.

PROPOSITION 3.23. *Let $g \in S_d$ be a homogeneous polynomial of degree d. Then the open subset $U = D(f) = \mathbb{P}^n \setminus Z(f)$ is affine, and every regular function on U can be written as g/f^r where $r > 0$ and g is a homogeneous polynomial of degree dr.*

PROOF. Let ϕ be the Veronese embedding of degree d; then U is isomorphic to the image of ϕ intersected with the complement of a hyperplane H in $\mathbb{P}^n$. Hence it is isomorphic to a closed subset of $\mathbb{P}^N \setminus H$, which is affine. The second statement follows from the fact that every regular function on U yields a regular function on its inverse image in $\mathbb{A}_0^{n+1}$, which is just $\mathbb{A}^n \setminus Z(f)$. Hence there exists an r and a polynomial g such that the function has the form g/f^r. Since it must be invariant under multiplication by λ, it follows that f must be homogeneous of the same degree as f^r. □

4. Projective subspaces

Let $V \subset \mathbb{P}^n$ be a closed subvariety defined as $Z(g_1, \dots, g_r)$ where the g_i are homogeneous polynomials of degree 1; we may assume that they are also linearly independent. Such a V is naturally isomorphic to $\mathbb{P}^{n-r}$ and si called a *projective* or *linear subspace* of $\mathbb{P}^n$, of dimension $(n-r)$ and codimension r. The cone over a projective subspace is a vector subspace of $\mathbb{A}^{n+1}$, and conversely. Note that a projective subspace of dimension (-1) is just the empty set, and one of dimension zero is a point. A projective subspace of dimension 1 is called a line; a projective subspace of codimension 1 is called a hyperplane.

LEMMA 3.24. *The set of hyperplanes in $\mathbb{P}^n$ is in natural bijection with $(\mathbb{P}^n)^\vee$, the dual projective space.*

EXERCISE 3.25. Assume that $P_0, \dots, P_r$ are points in $\mathbb{P}^n$ which are not contained in any subspace of dimension $r-1$. Then they are contained in a unique subspace of dimension r. In particular two distinct points are contained in a unique line.

EXERCISE 3.26. If $V, W \subset \mathbb{P}^n$ are projective subspaces of codimensions r and s, their intersection is a projective subspace of codimension $\leq r+s$; in particular if $r+s \leq n$, then $V \cap W$ is nonempty. In particular a line and a hyperplane always intersect, and if the line is not contained in the hyperplane they only intersect in one point.

DEFINITION 3.27. Let $p \in \mathbb{P}^n$ be a point and $H \subset \mathbb{P}^n$ a hyperplane with $p \notin H$. Define the projection with center p to be the map $\phi : \mathbb{P}^n \setminus \{p\} \to H$ as $\phi(q) = l(pq) \cap H$, where $l(pq)$ is the line containing p and q.

EXERCISE 3.28. Prove that the projection with center p is welldefined and a morphism; prove that if we replace H by some H', and define the corresponding projection $\phi' : \mathbb{P}^n \setminus \{p\} \to H'$, there is a unique isomorphism

$\lambda : H \to H'$ such that $\phi' = \lambda \circ \phi$. Hint: choose coordinates such that $p = (0, \ldots, 0, 1)$ and $H = \{x_n = 0\}$.

EXERCISE 3.29. Let $C_d \subset \mathbb{P}^d$ be the rational normal curve of degree d. Let $p \in C_d$, and H a hyperplane not containing p. Prove that the projection from center p induces an isomorphism between $C_d \setminus p$ and its image; prove that the isomorphism extends to an isomorphisms of the closures, and that the closure of the image is C_{d-1}.

DEFINITION 3.30. Let $X \subset \mathbb{P}^n$ be a closed subvariety. A projective cone over X with center p is a closed subvariety of $\mathbb{P}^{n+1}$ which is the closure of $\pi^{-1}(X)$, where $\pi : \mathbb{P}^{n+1} \setminus p \to \mathbb{P}^n$ is the projectioon from p and we choose an isomorphism of H with $\mathbb{P}^n$.

EXERCISE 3.31. Prove that the Veronese surface of degree 2 in $\mathbb{P}^5$ is not contained in an hyperplane. Prove that it can be isomorphically projected to a hyperplane. It is indeed the only surface in $\mathbb{P}^5$ with this property, but we will not prove this.

5. Hypersurfaces

DEFINITION 3.32. A *hypersurface* of degree d in $\mathbb{P}^n$ is a closed subvariety whose defining ideal is generated by a homogeneous polynomial of degree d. A hypersurface of degree 2 is called a quadric.

To do the following exercise, you will need to know that $K[x_0, \ldots, x_n]$ is a unique factorization domain. In particular, you should know the definition of UFD; see appendix.

EXERCISE 3.33. Let $f \in K[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d. Prove that $Z(f)$ is a hypersurface. Prove that it has degree d if and only if f does not have multiple factors. Prove that if X is a hypesurface of degree d its defining equation is unique up multiplication by a nonzero scalar.

DEFINITION 3.34. Let $f \in K[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree 2 and assume that $\operatorname{char} K \neq 2$; f defines a quadratic form, whose rank r we call the rank of the quadric. One has $2 \leq r \leq n+1$ (if $r = 1$ then f is the square of a linear form). If the rank is $n+1$ we call the quadric smooth.

EXERCISE 3.35. Prove that two quadrics of the same rank in $\mathbb{P}^n$ are isomorphic. In fact, the converse is also true as we will prove later.

EXERCISE 3.36. Prove that C_1 is a quadric of rank 3 in $\mathbb{P}^2$; deduce that every smooth quadric in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

EXERCISE 3.37. Prove that a cone over a hypersurface of degree d is a hypersurface of degree d. Prove that a quadric is a projective cone if and only if it is not smooth.

CHAPTER 4

Products

1. Products of algebraic varieties

We are all familiar with the notion of product; product of sets, but also of groups, vector spaces, topological spaces, and manifolds.

Definition 4.1. Let X and Y be spaces with K-functions. For any algebraic varieties A, B denote by $\mathrm{Mor}(A, B)$ the set of morphism from A to B[1]. A *product* of X and Y is a space with K functions Z together with projection morphisms $p : Z \to X$ and $q : Y \to Z$ such that, for every space with functions W, the natural map of sets

$$\mathrm{Mor}(W, Z) \longrightarrow \mathrm{Mor}(W, X) \times \mathrm{Mor}(W, Y)$$

given by $f \mapsto (p \circ f, q \circ f)$ is bijective.

Exercise 4.2. Prove that a product of X and Y, it it exists, is unique up to canonical isomorphism. In particular, prove that if you can give the *set* $X \times Y$ a structure of space with functions such that the two projections make it into a product for X and Y, then such a structure is unique.

Lemma 4.3. *Let $X = \mathbb{A}^n$ and $Y = \mathbb{A}^m$. Let $Z = \mathbb{A}^n \times \mathbb{A}^m$ and give it the structure defined by identifying the product with $\mathbb{A}^{n+m}$. Then Z is a product (hence, the product) of X and Y.*

Proof. An easy exercise, using that morphisms from any space with functions W to $\mathbb{A}^r$ are in bijection with $\mathcal{O}_W(W)^r$. □

Proposition 4.4. *Let X and Y be spaces with functions, and assume that $X \times Y$ has a product structure. Let $U \subset X$ and $V \subset Y$ be subspaces. Then the structure induced on $U \times V$ by $X \times Y$ makes $U \times V$ into a product for U and V.*

Proof. Just diagram checking, together with all the universal properties. □

Corollary 4.5. *Let U and V be affine varieties. Then $U \times V$ has a product structure, and is affine.*

Proof. Assume U is a closed subspace of $X = \mathbb{A}^n$, and V is a closed subspace of $Y = \mathbb{A}^m$. Then $U \times V$ is closed in $\mathbb{A}^{n+m}$. □

Theorem 4.6. *Let X and Y be algebraic varieties. Then $X \times Y$ has a structure of algebraic variety making it into a product for X and Y as spaces with K functions.*

[1] if this sounds like category theory to you, it's because it *is* category theory.

PROOF. Cover X with open affines $\{U_i\}_{i\in I}$ and Y with open affines $\{V_j\}_{j\in J}$. Then we have a natural structure of product on the subset $Z_{ij} := U_i \times V_j$ of $X \times Y$. On any intersection $Z_{ik} \cap Z_{lm}$ the two structures agree, and the intersection is open in both Z_{ij} and Z_{lm} (by uniqueness of the product structure andf the fact that it commutes wiuth the induced subspace structure). Hence the Z_{ij} glue to give a global structure of space with functions to $X \times Y$. It is then an exercise to check that this is really a product of X and Y, and the Z_{ij} provide a finite open cover. □

2. Products of quasiprojective varieties

Fix positive integers $m, n > 0$, and let $N = nm + n + m = (n+1)(m+1) - 1$. Define a morphism $f : \mathbb{P}^n \times \mathbb{P}^m \to proj^N$ via $f(x_i, y_j) = (x_i y_j)$.

THEOREM 4.7. *The morphism f is well defined and is an isomorphism with its image, which is a closed subset in $\mathbb{P}^N$. It is called the* Segre embedding.

PROOF. We can identify $\mathbb{P}^N$ with the projective space associated to $(n+1) \times (m+1)$ matrices. It is an easy exercise in linear algebra to prove that f is injective, and its image is the locus of rank one matrices; this proves that it is closed, since a matrix has rank one iff all the 2×2 minors vanish, and each minor is a homogeneous polynomial in the entries (of degree 2). To prove that it is an isomorphism with the image is a local statement in $\mathbb{P}^N$. So it is enough to check it in each coordinate chart, and by symmetry it is enough to look at the open set where the first coordinate is nonzero. Its inverse image in $\mathbb{P}^n \times \mathbb{P}^m$ is $U_0 \times V_0$, where $U_0 = \{x_0 \neq 0\}$ and $V_0 = \{y_0 \neq]0\}$. We can replace U_0 and V_0 with affine spaces by setting $x_0 = y_0 = 1$. Then the morphism f becomes $f(x, y) = (1, x_1, \ldots, x_n, y_1, \ldots, y_n, x_i y_j)$. This is a graph and we are done by the following exercise. □

EXERCISE 4.8. Let $X \subset \mathbb{A}^n$, and $f : X \to \mathbb{A}^m$ be a morphism. Let $g : X \to \mathbb{A}^{n+m}$ be defined by $g(x) = (x, f(x))$. prove that the image of g (the *graph* of f) is closed and that $g : X \to g(X)$ is an isomorphism.

In fact, the image of the Segre embedding is cut out by quadrics, i.e., its ideal is generated by elements of degree 2.

EXERCISE 4.9. Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ is a rank 4 quadric in $\mathbb{P}^3$. Prove that $\mathbb{P}^1 \times \mathbb{P}^n$ is cut out by quadrics.

COROLLARY 4.10. *The product of projective varieties is projective. The product of quasiprojective varieties is quasiprojective.*

We will now give an alternate description of the topology in $\mathbb{P}^n \times \mathbb{P}^m$. Let $f \in K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a monomial. We say that it is bihomogeneous of bidegree (d, e) if it is homogeneous of degree d in the x_i's and of degree e in the y_j's. We say that any polynomial is bihomogeneous of bidegree (d, e) if every monomial in it is so. The set of bihomogeneous polinomials of given bidegree is a subpace $S_{d,e}$ and $K[x_0, \ldots, x_n, y_0, \ldots, y_m] = \oplus S_{d,e}$. For example, $2x_1x_2^2y_1 - x_3^3y_2$ is bihomogeneous of bidegree $(3, 1)$.

The following will provide a key step in the proof that projective space is complete.

Proposition 4.11. *Let X be a closed subset of $\mathbb{A}_0^{n+1} \times \mathbb{A}^m \subset \mathbb{A}^{n+m+1}$. If X is the closure of the inverse image of a closed subset of $\mathbb{P}^n \times \mathbb{A}^m$, then $E(X)$ is generated by bihomogeneous elements. Conversely, the zero locus of any bihomogeneous ideal intersected with $\mathbb{A}_0^{n+1} \times \mathbb{A}^m$ is the inverse image of a closed subset of $\mathbb{P}^n \times \mathbb{P}^m$.*

Proof. The proof is analogous to the one for the Zariski topology of projective space. The thing to check is that the structure of quotient space with functions and of product commute with each other. □

3. Grassmann variety

Fix a vector space V of dimension n over K. V has a natural structure of algebraic variety, isomorphic to $\mathbb{A}^n$. For $0 < k < n$ let $G(k,V)$ be the set of all k-dimensional linear subspaces of V. Let $B(k,V) \subset V^k$ be the set of linearly independent k-tuples; it is open in V^k and hence itself an algebraic variety. As a set, $G(k,V)$ is the quotient of $B(k,V)$ by the natural $GL(k,K)$ action. We give it the induced structure of function space. It is easy to see that an isomorphism of linear spaces $V \to W$ induces an isomorphism $B(k,V) \to B(k,W)$ and hence $G(k,V) \to G(k,W)$.

Theorem 4.12. *$G(k,V)$ is an algebraic variety, with an open cover by affines isomorphic to $\mathbb{A}^{k(n-k)}$.*

Proof. We may assume that $V = K^n$, and identify $B(k,V)$ with the set of $(k \times n)$ matrices of maximal rank k, by associating to each $k$tuple the matrix which has the given vectors as columns. For any multiindex $I = (0 < i_1 < i_2 < \ldots < i_k \leq n)$ let $B_I \subset B(k,V)$ be the locus of tuples such that the I-th maximal minor of the matrix is invertible. The B_I are $GL(k)$ invariant, hence the inverse images of open subsets U_I of $G(k,V)$, and the U_I's cover $G(k,V)$. Of course they are all isomorphic, so let $U = U_{1,\ldots,k}$; it will be enough to prove that U is isomorphic to $\mathbb{A}^{k(n-k)}$. Let $Z \subset B(k,V)$ be the locus of matrices where the uppermost $k \times k$ minor is the identity. Clearly Z is isomorphic to $\mathbb{A}^{k(n-k)}$ (since whatever else we put in the matrix, it will have rank k). Modify the argument in the projective case to show that $Z \to U$ is an isomorphism of spaces with functions. □

Theorem 4.13. *$G(k,V)$ is a projective algebraic variety.*

Proof. Again assume that $V = K^n$. Define a morphism $B(k,V) \to \mathbb{A}_0^N$, where N is the cardinality is $\binom{n}{k}$, the cardinality of all possible multiindices I as in the proof of the previous theorem. Define a map $\phi : B(k,V) \to \mathbb{A}_0^N$ by sending a matrix to the determinants of all its maximal minors, in a fixed (e.g. lexicographical) order. Note that these map is a morpism, since determinants are polynomials, and its image doesn't meet zero since the matrix has rank k, hence at least one $k \times k$ minor must be invertible. ϕ induces a set map $\bar{\phi} : G(k,V) \to \mathbb{P}^{N-1}$, which must necessarily be a morphism. □

4. Blowup

Definition 4.14. Let $0 \in \mathbb{A}^n$. We define the *blowup* $B := Bl_0\mathbb{A}^n$ of $\mathbb{A}^n$ at the origin to be the closed algebraic subvariety of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ defined as

$Z(x_iy_j - x_jy_i)_{1\leq 1<j\leq n}$, where $x_1, \ldots, x_n$ are coordinates in $\mathbb{A}^n$ and $y_1, \ldots, y_n$ are coordinates in $\mathbb{P}^{n-1}$. It is closed since the given equations are homogeneous (of degree 1) in the variables y_j's. The projection of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ on the first factor induces a structure map $\varepsilon : B \to \mathbb{A}^n$. The inevrse image E of 0 is called *exceptional divisor* of the blowup.

THEOREM 4.15. *Let* $U = \mathbb{A}^n_0$, $U_i := D(y_i) \subset \mathbb{P}^n$, *and* $B_i := B \cap \mathbb{A}^n \times U_i$*; . Then:*

(1) $\varepsilon : \varepsilon^{-1}(U) \to (U)$ *is an isomorphism;*
(2) *the exceptional divisor* E *is isomorphic to* $\mathbb{P}^{n-1}$ *via the projection on the second factor;*
(3) *each* B_i *is isomorphic to* $\mathbb{A}^n$ *and* $aB_i \cap E$ *is a hyperplane.*

PROOF. (ii) is obvious, since every point of the form $(0, y)$ satisfies the equations trivially. (iii) We will prove it for $i = n$ to simplify notation. Let v_i be the regular function y_i/y_n on U_n, for $i = 1, \ldots, n$. Note that $v_n = 1$ and $v_1, \ldots, v_{n-1}$ are coordinates on U_n. Hence B_n is the closed subset of $\mathbb{A}^n \times U_n$ with equations $x_iv_j = x_jv_i$. Among these are $x_i = x_nv_i$ for $i < n$. Note conversely that a point (x, v) such that $x_i = x_nv_i$ is also in B_n. Therefore the regular functions $(v_1, \ldots, v_{n-1}, x_n$ define an isomorphism between U_n and $\mathbb{A}^n$. In such coordinates, the map ε is given by $\varepsilon(v_i, x_n) = (v_1x_n, \ldots, v_{n-1}x_n, x_n)$. It is immediate to check that $\varepsilon^{-1}(0) \cap B_n = B_n \cap Z(x_n)$. To prove (i), note that if $V := \varepsilon^{-1}(U)$ and $V_n := V \cap U_n$, then $\varepsilon : V_n \to \varepsilon(V_n)$ is an isomorphism, with inverse given by $(x_1, \ldots, x_n) \mapsto (x_1/x_n, \ldots, x_{n-1}/x_n, x_n)$. Since $\varepsilon(V_n) = D(x_n)$ and $V_n = \varepsilon^{-1}(D(x_n))$, statement (i) follows because the $D(x_n)$ are an open cover of $\mathbb{A}^n_0$. □

DEFINITION 4.16. Let X be a closed subvariety of $\mathbb{A}^n$; the *strict transform* $\hat{X}$ of X via the blowup ε is the closure of $\varepsilon^{-1}(X \cap \mathbb{A}^n_0)$ inside $Bl_0\mathbb{A}^n$. The exceptional divisor of the blowup is $\hat{X} \cap E$. We also call it the blowup of X at 0.

It is not difficult to check that blow up can be defined locally and then glued, so that if X is an algebraic variety and $p \in X$, it makes sense to talk about Bl_pX. See exercises. Blowup is related to rational maps, in fact it's the easiest example.

CHAPTER 5

Irreducibility and Noetherianity

1. Irreducibility

DEFINITION 5.1. A topological space is *reducible* if it is the union of two proper closed subsets. Otherwise it's *irreducible*.

EXERCISE 5.2. A topological space is irreducible if and only if every nonempty open subset is dense. A Hausdorff topological space is irreducible iff it is one point.

LEMMA 5.3. *For any topological space X, a subspace Y is irreducible iff its closure $\bar{Y}$ is.*

PROOF. Assume Y reducible. Write $Y = Y_1 \cup Y_2$ proper closed subsets. Then there exist closed subsets F_i in X such that $Y_i = F_i \cap Y$. We can assume that $F_i \subset \bar{Y}$ (otherwise replace it by $F_i \cap \bar{Y}$); moreover $F_i \neq \bar{Y}$ (otherwise $Y_i = Y$). Taking closures, $\bar{Y} = \bar{Y}_1 \times \bar{Y}_2 \subset F_1 \cup F_2$. Since the opposite inclusion has already been proven, we get a decomposition $\bar{Y} = F_1 \cup F_2$ of $\bar{Y}$ into proper closed subsets.
Conversely, assume $\bar{Y} = F_1 \cup F_2$ proper closed subsets. Let $Y_i = F_i \cap Y$; then $Y = Y_1 \cup Y_2$. If, say, $Y_1 = Y$ then $Y \subset F_1$ hence $\bar{Y} \subset F_1$, a contradiction. □

LEMMA 5.4. *A closed subset X of $\mathbb{A}^N$ is reducible iff there exist $f, g \in K[x_1, \ldots, x_n]$ such that $f, g \notin E(X)$ but $fg \in E(X)$.*

PROOF. Easy and left to the reader. □

DEFINITION 5.5. An ideal I in a ring A is *prime*[1] if it is proper and for any $a, b \in A$ the fact that $ab \in I$ implies that either a or b is in I.

It is easy to see that an ideal I is prime iff A/I is a domain. Every prime ideal is radical, and every maximal ideal is prime.

COROLLARY 5.6. *A closed subset X of $\mathbb{A}^n$ is irreducible if and only if $E(X)$ is prime.*

EXERCISE 5.7. The only Hausdorff subsets of $\mathbb{A}^1$ are the finite ones.

EXERCISE 5.8. Prove that the following algebraic varieties are irreducible: $\mathbb{A}^n$; $\mathbb{P}^n$; $\mathbb{P}^n \times \mathbb{P}^m$; $G(k, V)$.

EXERCISE 5.9. Prove that every irreducible topological subspace is connected, and give an example of a connected algebraic variety which is not irreducible. Prove that the image of an irreducible topological space under a continuous map is irreducible.

[1] See also appendix.

2. Noetherianity of rings and modules

DEFINITION 5.10. A ring is *Noetherian* if every ideal is finitely generated. Equivalently, if it satisfies the following ascending chain condition (acc) on ideals: every chain of ideals $I_1 \subset I_2 \subset \dots$ stabilizes, i.e. there exists n_0 such that $I_n = I_{n_0}$ for all $n \geq n_0$.

EXERCISE 5.11. If K is a field, $K[x]$ is a principal ideal domain (PID), hence Noetherian. A principal ideal domain is a domain in which every ideal is principal.

THEOREM 5.12 (Hilbert basis Theorem). *If A is Noetherian then $A[x]$ is also Noetherian.*

PROOF. Assume $I \subset A[x]$ is not a finitely generated ideal. Let $f_1 \subset I$ be a nonzero element of minimal degree d_1 in x. Choose inductively f_n of minimal degree d_n in $I \setminus I_{n-1} := (f_1, \dots, f_{n-1})$ and let a_i be the leading coefficient of f_i. Let $J_i = (a_1, \dots, a_i) \subset A$; since $I_i \subset I_{i+1}$ the sequence stabilizes, say at I_N. Therefore $a_{N+1} = \sum_{i=1}^N b_i a_i$ for some $b_i \in A$. Hence the polynomail

$$g = \sum_{i=1}^{N} b_i f_i x^{d_{N+1} - d_i}$$

is in I_N and has the same degree and leading coefficient as f_{N+1}; therefore $f_{N+1} - g$ has degree strictly smaller then d_{N+1} and is not in I_N, contradicting the minimality of d_{N+1}. □

COROLLARY 5.13. *The ring $K[x_1, \dots, x_n]$ is noetherian.*

EXERCISE 5.14. If a ring A is Noetherian and $f : A \to B$ is a surjective homomorphism, then B is Noetherian. In particular, every finitely generated K algebra is Noetherian.

DEFINITION 5.15. Let A be a ring and M an A module. M is called *Noetherian* if every submodule is finitely generated; equivalently, if every ascending sequence of submodules stabilizes.

REMARK 5.16. A ring A is Noetherian if and only if it is Noetherian as A-module.

EXERCISE 5.17. Let A be a ring, and $f : M \to N$ a surjective homomorphism of A-modules. If M is Noetherian, then N is also Noetherian, and so is every submodule of M.

THEOREM 5.18. *If A is a Noetherian ring, then an A-module M is Noetherian iff it is finitely generated.*

PROOF. $\Rightarrow$ is trivial; every Noetherian module is by definition finitely generated. $\Leftarrow$. It is enough to prove that A^n is Noetherian; we do this by induction on n, the case $n = 1$ being trivial. Let $\pi : A^n \to A$ be the projection on the last factor, so that $\ker \pi$ is isomorphic to A^{n-1}. Let N be a submmodule of A^n, $N' = \pi(N)$ and $N'' = N \cap \ker \pi$. Then N' and N'' are Noetherian by induction, so they are finitely generated by elements $n'_1, \dots, n'_k$ and $n_{k+1}, \dots, n_r$ respectively. Let $n_i \in N$ be elements such that $\pi(n_i) = n'_i$. Then $n_1, \dots, n_r$ generate N. □

EXERCISE 5.19. If S is a multiplicative part of A, then $S^{-1}(A)$ is Noetherian. If M is a finitely generated A module, then $S^{-1}M$ is a finitely generated $S^{-1}A$ module.

3. Noetherianity for topological spaces

DEFINITION 5.20. A topological space is *Noetherian* if every descending sequence of closed subsets stabilizes.

PROPOSITION 5.21. *Let X be a topological space. The following are equivalent:*

(1) *X is a Noetherian topological space;*
(2) *Noetherian induction holds; i.e., any nonempty collection $\mathcal{F}$ of closed subsets of X contains a minimal*[2] *element F.*
(3) *every open subset is compact*[3].

PROOF. (i) implies (ii). Let A be a nonempty collection of closed subsets without a minimal element. Choose F_0 in A; since it is not minimal, one can find $F_1 \subset F_0$ a proper closed subset which is also in A. Repeating this argument yields an infinite strictly decrasing sequence of closed subsets, against Noetherianity.
(ii) implies (i). Let F_i be a descending sequence of closed subsets, and apply Noetherian induction to the set $\{F_i\}$.
(i)+(ii) imply (iii). Let U be an open subset, $\{U_i\}_{i\in I}$ an open cover of U. Let F_i be the complement of U_i.Choose $i_0 \in I$ such that F_{i_0} is a minimal element of the set $\{F_i\}$, and consider the collection $\{F_i^1\} = \{F_i \cap F_{i_0}\}$. Choose $i_1 \in I$ such that $F_{i_1}^1$ is a minimal element, and consider the collection $\{F_i^2\} = \{Fi_0 \cap F_{i_1} \cap F_i\}$; choose i_2 such that $F_{i_2}^2$ is a minimal element, and so on. This gives a descending sequence of closed subsets $F_{i_0} \supset F_{i_1}^1 \supset \ldots$ which must stabilize, say at $F_{i_n}^n$. Therefore, for each $i \in I$, $F_i \supset F_{i_0} \cap F_{i_1} \cap \ldots \cap F_{i_n}$, or in other words $U_i \subset U_{i_0} \cup \ldots \cup U_{i_n}$, hence $\{U_{i_j}\}_{j=0}^n$ is a finite subcover of U.
(iii) implies (i). Let F_i be a descending sequence of closed sets, $F = \bigcap F_i$. Then F is closed, its complement U is open, and the $U_i := X \setminus F_i$ are an open cover of it, with $U_0 \subset U_1 \subset \ldots$. Assume $U_0, \ldots, U_n$ is a finite subcover; then $U = U_n$ and we are done. □

EXERCISE 5.22. Prove that if a topological space is Noetherian, then every subspace and every continuous image is also Noetherian.

PROPOSITION 5.23. *If a topological space has a finite cover by Noetherian subspaces, then it is also Noetherian.*

PROOF. Let $F_1 \subset F_2 \subset \ldots$ be an ascending sequence of closed subsets in X. Then for each i the sequence $F_n \cap U_i$ is an ascending sequence of closed subsets in U_i, therefore it stabilizes, say at n_i. Let $N = \max\{n_i\}$. Then F_n stabilizes at N (why?). □

THEOREM 5.24. *Affine space $\mathbb{A}^n$ is Noetherian.*

[2]one such that $F \in \mathcal{F}$ and $F' \notin \mathcal{F}$ for every proper closed subset F' of F.

[3]A topological space is compact if every open cover has a finite subcover. Some people call this quasicompact.

PROOF. A descending sequence of closed subsets F_i induces an anscending sequence of ideals $E(F_i)$ in $K[x_1,\ldots,x_n]$. This stabilizes by Noetherianity of the ring $K[x_1,\ldots,x_n]$. □

COROLLARY 5.25. *Every algebraic variety is Noetherian.*

PROOF. Every affine variety is Noetherian, because it's a subspace of some affine space; and every algebraic variety has a finite cover by affine varieties. □

PROPOSITION 5.26. *Every Noetherian topological space X is the union of finitely many irreducible closed subsets.*

PROOF. Let $\mathcal{F}$ be the collection of all closed subsets of X which are not the union of finitely many irreducible closed subsets. If $\mathcal{F}$ is not empty, it has a minimal element F. F cannot be irreducible: let $F = F_1 \cup F_2$ be a decomposition of F in proper closed subsets. By minimality neither $F_i \in \mathcal{F}$, hence there exist (for $i = 1, 2$) n_i irreducible closed subsets $F_{i1},\ldots,F_{in_i}$ such that

$$F_i = \bigcup_{j=1}^{n_i} F_{in_j}.$$

Therefore

$$F = \bigcup_{j=1}^{n_1} F_{1n_j} \cup \bigcup_{j=1}^{n_2} F_{2n_j}$$

is a finite union of irreducibles, contradicting $F \in \mathcal{F}$. □

EXERCISE 5.27. Let X be a topological space, G an irreducible closed subset, $F_1,\ldots,F_n$ closed subsets. If $G \subset F_1 \cup \ldots \cup F_n$, then there is i such that $G \subset F_i$. We will use this in the theorem below, so prove it!

DEFINITION 5.28. Let X be a topological space. An irreducible component of X is a maximal irreducible closed subset.

THEOREM 5.29. *Every Noetherian topological space X is the union of its irreducible components, that are a finite number.*

PROOF. Let $X = X_1 \cup \ldots X_n$ be a decomposition of X into irreducible closed subsets. We may assume that no X_i is contained in another X_j. Then each X_i is an irreducible component; in fact, if $X_i \subset F$ with F closed irreducible, we must have $F \subset \bigcup X_i$ while $X \not\supset X_j$ for any j, a contradiction. Moreover, each irreducible component is one of the X_i's. In fact, if Y is an irreducible component of X, since $Y \subset \bigcup X_i$, there exists i such that $Y \subset X_i$; since Y is maximal irreducible, necessarily $Y = X_i$. □

THEOREM 5.30. *If X and Y are irreducible algebraic varieties, their product is also irreducible.*

PROOF. Assume that $X \times Y = Z \cup W$ where Z and W are closed. For $x \in X$, $y \in Y$ write ${}_xY := \{x\} \times Y$ and ${}_yX := X \times \{y\}$. Note that ${}_xY$ is isomorphic to Y and ${}_yX$ is isomorphic to X, hence they are irreducible. Since ${}_xY = ({}_xY \cap Z) \cup ({}_xY \cap W)$, we must have either ${}_xY \subset Z$ or ${}_xY \subset W$ (or both). Let $A := \{x \in X \mid {}_xY \subset Z\}$ and $B = \{x \in X \mid {}_xY \subset W\}$; one has $X = A \cup B$. If we prove that A and B are closed we are done, because then one of them must be equal to X. For all $y \in Y$ one has that $A \times \{y\} \subset {}_yX \cap Z$ which is closed in ${}_yX$, hence $\bar{A} \times \{y\} \subset {}_yX \cap Z \subset Z$. Since y was arbitrary, it follows that $\bar{A} \times Y \subset Z$, hence that $A = \bar{A}$. $\square$

EXERCISE 5.31. If a topological space is Hausdorff, then its largest irreducible components are the points. Find the irreducible components of $Z(xy, xz) \subset \mathbb{A}^3$. Draw a picture.

The following is easy but extremely important, so do it. Look up the section on UFD's in the appendix, if needed.

EXERCISE 5.32. A principal ideal (f) in $K[x_1, \ldots, x_n]$ is prime if and only if f is irreducible. In general, the irreducible components of $Z(f)$ are the zero loci of the irreducible factors of f.

CHAPTER 6

Topological properties of algebraic varieties

1. Separatedness

Definition 6.1. Let X be a set. The diagonal of X, Δ_X, is the subset of $X \times X$ which is the image of X via the map $\delta(x) = (x, x)$. Note that if X is an algebraic variety, then δ is a morphism.

Exercise 6.2. Let X be a topological space. Prove that X is Hausdorff iff Δ_X is closed in $X \times X$ (where the latter has the product topology).

Definition 6.3. An algebraic variety is *separated* if the diagonal is closed in the product of the variety with itself.

Lemma 6.4. *If X is a separated algebraic variety, and $Y \subset X$ is a locally closed subvariety, then Y is also separated.*

Proof. The diagonal Δ_Y is equal to $Y \times Y \cap \Delta_X$ in $X \times X$. Since taking products commutes with taking subspaces, the topology on $Y \times Y$ is the induced topology of $X \times X$; therefore, since Δ_X is closed in $X \times X$, the result follows. □

Theorem 6.5. *Projective space $\mathbb{P}^n$ is separated.*

Proof. Consider $\mathbb{P}^n \times \mathbb{P}^n$ as a closed subvariety of $\mathbb{P}^N$ where $N = n^2 - 1$ and $\mathbb{A}_0^{N+1}$ is the space of nonzero $n \times n$ matrices. It is an easy exercise in linear algebra to verify that $\Delta_{\mathbb{P}^n}$ is the intersection on $\mathbb{P}^n \times \mathbb{P}^n$ with the linear subspace of symmetric matrices. Therefore it is closed. □

Corollary 6.6. *Every affine, projective and quasiprojective variety is separated.*

Corollary 6.7. *If X is any algebraic variety, then Δ_X is locally closed in $X \times X$.*

Proof. Let $p \in X$, and let U be an affine open neighborhood of p. Then $(p, p) \in \Delta_U = \Delta_X \cap (U \times U)$. Since U is affine, it is separated, hence $\Delta_X \cap V$ is open in $V := U \times U$, which is an open neighborhood of (p, p) in $X \times X$. □

Exercise 6.8. The affine line with the origin doubled is nonseparated. The usual construction of a space which satisfies all the axioms of being a manifold except Hausdorffness can be easily adapted to produce a nonseparated algebraic variety. Let $X = Z(y^2 - y) \subset \mathbb{A}^2$. Let Y be its quotient space with functions by the equivalence relation $(x, y) \sim (x', y')$ iff $x = x'$ and either $y = y' = 0$ or $yy' \neq 0$. Prove that Y is a nonseparated algebraic variety.

Our terminology in this notes is not completely consistent with the literature, where most often a variety is supposed to be separated and irreducible. Moreover, in many texts a variety is directly assumed to be quasiprojective. In fact, the category of algebraic varieties as defined here is equivalent to that of reduced schemes of finite type over K (via the functor t of Hartshorne, Proposition II.2.6).

2. Completeness

Definition 6.9. An algebraic variety is *complete* if it is separated and moreover for every algebraic variety Y the natural projection $\pi : X \times Y \to Y$ is closed, i.e., maps closed subsets to closed subsets.

In view of the fact that the product of algebraic varieties is not the topological product, this is a good substitute of the notion of compactness, as seen by the following remark.

Remark 6.10. Let X be a topological space. If X is compact (in the sense that every open cover admits a finite subcover) then for every topological space Y the projection $X \times Y \to Y$ is closed. Moreover, if X is a noncompact topological space with a countable basis for the topology, then the natural projection $X \times \mathbb{R} \to \mathbb{R}$ is not closed.

In fact, complete algebraic varieties share many properties of compact spaces.

Definition 6.11. Let $F : X \to Y$ be a morphism of algebraic varieties. The *graph* of f is

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\} = h^{-1}(\Delta_Y)$$

where $h : X \times Y \to Y \times Y$ is the morphism $h(x, y) = (f(x), y)$. The graph is locally closed in $X \times Y$, and closed if Y is separated; it is therefore also an algebraic variety.

Remark 6.12. Let $f : X \to Y$ be a morphism of algebraic varieties. Then Γ_f is isomorphic to X via the projection to the first factor.

Corollary 6.13. *Let X be a complete algebraic variety, Y a separated algebraic variety and $f : X \to Y$ a morphism. Then $f(X)$ is closed in Y.*

The following is an analogous of the fact that compact complex manifolds have no nonconstant global holomorphic functions.

Corollary 6.14. *Let X be a complete connected algebraic variety. Then every regular function on X is constant.*

Proof. A regular function $f \in \mathcal{O}_X(X)$ is the same as a morphism $f : X \to \mathbb{A}^1$. Identify $\mathbb{A}^1$ as an open subset of $\mathbb{P}^1$. Therefore $f(X)$ is a closed subset of $\mathbb{P}^1$ which is contained in $\mathbb{A}^1$, hence is not all of $\mathbb{P}^1$. It must be a finite set. Since it is connected, it must be a point. □

Remark 6.15. Let X be a complete algebraic variety. A subvariety Y of X is complete if and only if it is closed.

Theorem 6.16. *For every $n > 0$ projective n-space $\mathbb{P}^n$ is a complete algebraic variety.*

PROOF. We have to show that for every variety Y, the projection $q : \mathbb{P}^n \times Y \to Y$ is closed. We first reduce to assuming Y affine, and then to the case where Y is $\mathbb{A}^m$. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be a closed subset. Then $E(Z)$ is generated by a finite number of polynomials $f_1, \ldots, F_r \in K[x_0, \ldots, x_n, y_0, \ldots, y_m]$, with f_i homogeneous of degree d_i in the projective variables. For $y \in \mathbb{A}^m$, let $I(y)$ be the ideal in $K[x_0, \ldots, x_n]$ generated by $f_i(y)$. By projective NSS, $y \in q(Z)$ iff $Y(y)$ does not contains $S_N := K[x_0, \ldots, x_n]_N$ for any N.
So we need only prove that $U_N = \{y \mid S_N \subset I(y)\}$ is open. Consider the linear map

$$\phi(y) : \oplus_{i=1}^r S_{N-d)i} \to S_N \qquad g_i \mapsto \sum_{i=1}^r f_i(y) g_i.$$

Fixing bases of each S_d, we can view it as a matrix with polynomial entries in the y variables. But U_N is the locus where $\phi(y)$ is surjective, hence where it has maximal rank, and this is open (by the connection between determinant and rank, and the fact that the determinant is a polynomial). □

COROLLARY 6.17. *A quasiprojective algebraic variety is complete if and only if it is projective.*

PROOF. If X is projective, then it is closed in some projective space, hence complete. If X is quasiprojective and complete, there exists a morphism $f : X \to \mathbb{P}^N$ such that $f(X)$ is locally closed in $\mathbb{P}^N$ and $f : X \to f(X)$ is an isomorphism. Since $\mathbb{P}^N$ is separated, $f(X)$ is closed, and so X is projective. □

EXERCISE 6.18. By the corollary, we know that $\mathbb{A}^1$ is not complete. Show directly that $q : \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ (i.e., the morphism $\mathbb{A}^2 \to \mathbb{A}^1$ given by $(x, y) \mapsto x$) is not closed.

3. Fibered products

DEFINITION 6.19. Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of algebraic varieties. Let

$$X \times_Y Z := \{(x, z) \in X \times Z \mid f(x) = g(z)\} = h^{-1}(\Delta_Y)$$

where $h : X \times Z \to Y \times Y$ is the morphism $h(x, z) = (f(x), g(z))$. The variety $X \times_Y Z$ is called *fibered product*[1] of X and Z over Y.

PROPOSITION 6.20. *The fibered product is the parent of all commutative diagrams with the same right-bottom corner; i.e., given an algebraic variety W and morphisms $a_1 : W \to X$ and $a_2 : W \to Z$ such that $f \circ a_1 = g \circ a_2$, there exists a unique morphism $a : W \to X \times_Y Z$ such that $a_i = p_i \circ f$, for $i = 1, 2$.*

PROOF. Just sit down and check it for yourself. It's easy. □

Note that the proposition could be used as a definition of fibered product up to canonical isomorphism.

[1] It's either fibre or fiber, hence either fibred or fibered, depending on whether it's UK or US english. We really should go back to writing algebraic geometry in french.

Some special cases are worth mentioning. If $g : Z \to Y$ is the inclusion of a locally closed subvariety, then $X \times_Y Z = f^{-1}(Z)$. If in particular $Z = \{y\}$ is a point in Y, then $X \times_Y \{y\} = X_y = f^{-1}(y)$ is the fiber of f over y. In general, the fiber of $p_1 : X \times_Y Z \to X$ over x is isomorphic via p_2 to the fiber $Z_{f(x)}$ of g.

Despite the really simple definition, fibered products are terribly important in algebraic geometry, and they often reappear under other names, like pullback (popular for bundles and sometimes fibrations) and base change (usually used for flat families). Note that products do exist for manifolds, but fibered products in general do not. Fibered products exist for topological spaces but as far as I know are not terribly useful.

The fibered product of algebraic varieties *as varieties* coincides with their fibered product *as spaces with functions*, but it may be different from their fibered product *as schemes*.

CHAPTER 7

Dimension theory

1. Definition of dimension

DEFINITION 7.1. Let X be a nonempty topological space. A *length n chain* in X is a sequence

$$\emptyset \neq X_0 \subset X_1 \subset \ldots \subset X_n \subseteq X$$

where each X_i is a close irreducible subset and $\subset$ means strict inclusion. The *dimension of* X is the supremum of the lengths of chians in X. It is negative if X is empty, otherwise it can be finite or ∞.

EXAMPLE 7.2. Let X be a topological space. If $U \subset X$, then $\dim U \leq \dim X$. If $Y \subset X$ is a proper closed subset and X is irreducible, then $\dim Y < \dim X$.

LEMMA 7.3. *If X has a finite open cover by open subsets X_i, then $\dim X = \max \dim X_i$.*

EXAMPLE 7.4. An algebraic variety has dimension zero if and only if it is a disjoint union of points. The dimension of a variety is the supremum of the dimension of its irreducible components. The dimension of $\mathbb{A}^1$ and $\mathbb{P}^1$ is 1. The dimension of $\mathbb{P}^n$ and $\mathbb{A}^n$ is at least n.

2. Finite morphisms

Recall that if X is an affine variety, we write $K[X]$ for $\mathcal{O}_X(X)$.

DEFINITION 7.5. A morphism $f : X \to Y$ is *affine* if there is an affine open cover U_i of the image such that each $V_i := f^{-1}(U_i)$ is also affine. It is *finite* if we can choose such U_i so that $K[V_i]$ is a finitely generated $K[U_i]$ module.

REMARK 7.6. (i) Let $f : X \to Y$ be a finite morphism. Let $V \subset Y$ be an open subset, and $U = f^{-1}(V)$; then $f : V \to U$ is also finite. (ii) Every closed embedding is finite. (iii) A composition of finite morphisms is finite. (iv) If $f \in \mathcal{O}_X(X)$ is regular, then the inclusion of the open subset where f is nonzero is affine.

PROOF. (i) We can assume that Y is affine, X is affine, and $K[X]$ is a finite $K[Y]$ module. If $V = D(f)$ is principal, then $K[V] = K[Y]_f$ and $K[U] = K[X]_f$, hence $K[U]$ is a finite $K[V]$ module (with the same generators as $K[X]$ as $K[Y]$ module). The general case follows since principal open subsets are a basis of the topology.
(ii) Let Y be an affine variety, X a closed subvariety. Then X is affine and $K[Y] \to K[X]$ is surjective, therefore $K[X]$ is generated by 1 as a $K[Y]$

module.
(iii) This follows by Proposition A.11 and (i).
(iv) We will not need this result, so we leave the easy proof to the reader. □

LEMMA 7.7. *If* $\phi : X \to Y$ *is a finite morphism of affine varieties, then* $K[X]$ *is a finite* $K[Y]$ *module.*

PROOF. Let $f \in K[X]$. Find an open cover by principal open subsets $D(g_i)$ of Y such that $K[X]_{g_i}$ is a finite $K[Y]_{g_i}$ module. For each i we can find a polynomial $F_i \in K[Y][t]$ such that the coefficient of the top degree term is a power h_i of g_i, with $F_i(f) = 0$. We may assume all F_i to have the same degree (multiply by a power of t if necessary).
Since $\cap Z(g_i) = \emptyset$, also $\cap Z(h_i) = \emptyset$, hence the h_i generate $K[Y]$ as an ideal. Choose $a_i \in K[Y]$ such that $\sum_i a_i h_i = 1$. Then $F = \sum a_i F_i \in K[Y][t]$ is a monic polynomial such that $F(f) = 0$. □

PROPOSITION 7.8. *If* $f : X \to Y$ *is a finite morphism, then it is closed.*

PROOF. We can ssume that $f(X)$ is dense in Y and prove that f is surjective. We can also assume that X and Y are affine and $K[X]$ is a finite $K[Y]$-module. Let $p \in Y$. Then the inclusion $p \to Y$ corresponds to a homomorphism $v_p : K[Y] \to K$ (namely $v_p(g) = g(p)$). The fact that $f(X)$ is dense in Y can be expressed by saying that $f^* : K[Y] \to K[X]$ is injective. Hence v_p extends by CorollaryA.18 to a homorphism $v_q : K[X] \to K$ which in turn corresponds to a point q in X, whose image via f must be p. □

LEMMA 7.9. *Let* $f : X \to Y$ *be a finite morphism, and* $Z \subset X$ *a closed subset; let* $W = f(Z)$. *Then* $f : Z \to W$ *is also finite.*

PROOF. We may assume that this is all affine. Then $f^{-1}(W) \to W$ is finite (check this!) and $Z \to f^{-1}(W)$ is a closed embedding. □

PROPOSITION 7.10. *Let* X *be a hypersurface in* $\mathbb{A}^n$; *then there exists a linear projection* $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ *such that* $\pi : X \to \mathbb{A}^{n-1}$ *is finite and surjective.*

PROPOSITION 7.11. *Assume that* $f : X \to Y$ *is finite, and let* $Z \subset W$ *be a strict inclusion of closed subsets in* X, *with* W *irreducible. Then* $f(Z) \neq f(W)$.

PROOF. We may assume that $W = X$ and that $Y = f(W)$. Let $g \in K[X]$ such that $Z \subset Z(g) \neq X$. Note that $Z(g) = X$ iff $g = 0$ by NSS since $K[X]$ is prime. Let $H \in K[Y][t]$ be a monic polynomial with $H(g) \neq 0$, $H(t) = t^n + \sum a_i t^i$. We can assume that $a_0 \neq 0$ since $K[Y]$ is a domain. Then $f(Z) \subset Z(a_0) \neq Y$. □

COROLLARY 7.12. *A finite morphism has finite fibers.*

PROOF. Assume that F is a fiber. If $\dim F = 0$ (see below), we are done. Else, F contains an irreducible closed subset W which is not a point. Let $p \in Y$, and $Z = \{p\}$. Then $f(Z) = f(W)$, a contradiction. □

THEOREM 7.13. *Let* $f : X \to Y$ *be a finite surjective morphism of algebraic varieties. Then every chain* Y_i *in* Y *can be lifted to* X, *i.e. there exists a chain* X_i *in* X *with* $\pi(X_i) = Y_i$. *Conversely, given a chain* X_i *in* X, *its image is a chain in* Y.

PROOF. Let X_i be a chain in X. Since f is a finite, $f(X_i)$ is closed; it is anyway irreducible, and $f(X_i) \neq f(X_{i+1})$ by Lemma 7.11.
Conversely, let Y_i be a length n chain in Y. Then $f^{-1}(Y_n)$ is a finite union of irreducible components A_j, and $Y_n = \cup f(A_j)$. Since Y_n is irreducible and $f(A_j)$ is closed, there must be one of them whose image is Y_n. Call it X_n. Now consider the finite map $f : X_n \to Y_n$ and repeat the argument to construct $X_{n-1} \subset X_n$ closed irreducible such that $f(X_{n-1}) = Y_{n-1}$, and so on. This way one can lift Y_i to a length n chain in X. □

3. Noether normalization and dimension

It is easy to see that we can restate Noether's normalization as follows.

THEOREM 7.14. *Noether normalization lemma. Let X be an affine variety; then there exists $m > 0$ such that there is a finite surjective morphism $f : X \to \mathbb{A}^m$.*

PROOF. Let $A = K[X]$, and $B \subset A$ a subalgebra isomorphic to $K[\mathbb{A}^m]$ such that A is a finitely generated B-module. Then the inclusion $B \to A$ corresponds to a morphism $X \to \mathbb{A}^m$ which is clearly affine. Its image is dense, since the induced map on coordinate rings is injective. However its image is also closed, so it's surjective. □

In fact, looking at the proof, more can be said.

THEOREM 7.15. *(Noether normalization, geometric version) Let $X \subset \mathbb{A}^n$ be a closed subvariety. Then the projection $\pi : \mathbb{A}^n \to \mathbb{A}^{n-1}$ given by $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$ induces a finite map $X \to \mathbb{A}^{n-1}$ if $(0, \ldots, 0, 1)$ is not contained in the projective closure of X, where $\mathbb{A}^n$ is identified with the open subsets $U_0 = \{y_0 \neq 0\}$ of $\mathbb{P}^n$.*

COROLLARY 7.16. *Let $X \subset \mathbb{P}^m$ be a projective variety. Let $p \in \mathbb{P}^m \setminus X$, and let $H \subset \mathbb{P}^m$ be a hyperplane not containing p. Then the projection $\mathbb{P}^m \setminus \{p\} \to H$ induces a finite map $X \to H$.*

THEOREM 7.17. *Let X be a irreducible hypersurface in $\mathbb{A}^n$. Then $\dim X = n - 1$ and $\dim \mathbb{A}^n = n$.*

PROOF. Induction on n, the case $n = 0, 1$ is already done. There is a finite surjective[1] map $\pi : X \to \mathbb{A}^{n-1}$, hence $\dim X = n - 1$. Let X_i be a length m chain in $\mathbb{A}^n$; we need to prove that $m \leq n$. We can assume that $X_m = \mathbb{A}^n$ and X_{m-1} is a hypersurface. Then $m - 1 \leq n - 1$, and we are done. □

COROLLARY 7.18. *The dimension of $\mathbb{P}^n$ is n. The dimension of $G(k, V) = k(\dim V - k)$. The dimension of $\mathbb{P}^n \times \mathbb{P}^m$ is $m + n$. If X is an algebraic variety which is not a point and $x \in X$, then $\dim Bl_x X = \dim X$.*

COROLLARY 7.19. *Every algebraic variety has finite dimension.*

COROLLARY 7.20. *Let X be an irreducible affine variety of dimension m, U an open subset. Then $\dim U = m$.*

[1]check that it is surjective! Hint: prove that $K[x_1, \ldots, x_{n-1}] \to K[X]$ is injective.

Proof. If $X = \mathbb{A}^m$, it is enough to choose a length m chain of affine subspaces starting with a point in U. Let $\pi : X \to \mathbb{A}^m$ be the linear projection. Let $Y = X \setminus U$: $Z = \pi(Y)$ is a proper closed subset, let its complement be A. Then $\dim A = m$, and $\dim \pi^{-1}(A) = m$. But $\pi^{-1}(A) \subset U$, hence $\dim U \geq m$. □

Theorem 7.21. *Let X be an irreducible algebraic variety, U an open subset. Then* $\dim X = \dim U$.

Proof. Follows easily. Fill in the details for yourself. □

If $Z \subset X$ is closed and X is irreducible, then $\dim Z = \dim X$ implies $Z = X$.

Theorem 7.22. *Let X and Y in $\mathbb{A}^n$ be irreducible closed subvarieties. Then if $X \cap Y$ is nonempty, every one of its irreducible components has dimension at least* $\dim X + \dim Y - n$.

Proof. It is enough to check that $X \cap Y$ is naturally isomorphic to $X \times Y \cap \Delta_{\mathbb{A}^n}$ and then apply the previous corollary, noting that $\Delta_{\mathbb{A}^n} = Z(x_1 - y_1, \ldots, x_n - y_n)$ in $\mathbb{A}^n \times \mathbb{A}^n$. □

CHAPTER 8

More dimension theory

1. Krull's principal ideal theorem and applications

The main result in this section is a special case of Krull's principal ideal theorem (Hauptidealsatz).

THEOREM 8.1. *Let X be an irreducible affine variety of dimension n, and $f \in K[X]$ a nonzero function. Then every irreducible component of $Z(f)$ has dimension $n-1$.*

PROOF. If $X = \mathbb{A}^n$, we already know this.
For the general case, note that it is enough to assume that $Z(f)$ is irreducible. Otherwise, write $Z(f) = Z \cup W$ where Z is an irreducible component and W is the union of all other components. Then there exists $g \in K[X]$ such that $W \subset Z(g)$ and $Z \not\subset Z(g)$. Let $U = D(g)$; U is affine and irreducible, it has dimension n, and $Z(f|_U) = Z \cap U$ which has the same dimension as Z and is irreducible.
Let now $\pi : X \to \mathbb{A}^n$ be a finite surjective map. Then since $K[X]$ is a finite A-algebra, where $A = K[x_1, \dots, x_n]$, we can find a monic polynomial $h \in A[x_{n+1}]$ such that $h(f) = 0$. Since $A[x_{n+1}]$ is a UFD, we can factor h as product of irreducible polynomials, which must all be monis, and since X is irreducible f must be a root of one of them. So we may assume h is irreducible.
Define a morphism $\psi : X \to \mathbb{A}^{n+1}$ given by $\psi(p) = (\pi(p), f(p))$. ψ is finite, so $\psi(X)$ has dimension n. On the other hand $\psi(X) \subset Z(h)$ and the latter is irreducible of dimension n; so they are equal. Since ψ is finite, $\psi(Z(f))$ has the same dimension as $Z(f)$; but $\psi(Z(f)) = Z(h_0) \cap Z(x_{n+1}) \subset \mathbb{A}^{n+1}$ where h_0 is the constant term of h, as one can easily verify. Therefore it has dimension $n-1$. □

COROLLARY 8.2. *Let X be an irreducible affine variety of dimension n, and $f_1, \dots, f_r \in K[X]$. Let $Y = Z(f_1, \dots, f_r)$. Then if Y is nonempty, every one of its irreducible components has dimension at least $n-r$.*

PROOF. Induction on r and repeated application of the previous theorem. □

COROLLARY 8.3. *Let X be an irreducible affine variety of dimension n, and $\pi : X \to \mathbb{A}^n$ a finite surjective morphism. Then if $Z \subset \mathbb{A}^n$ is a d-dimensional affine subspace, every irreducible component of $\pi^{-1}(Z)$ has dimension d.*

PROOF. Let W be an irreducible component of $\pi^{-1}(Z)$. Then $\dim W = \dim \pi(W)$ since π is finite, and $\pi(W) \subseteq Z$ so that $\dim W \leq \dim Z$. On the other hand we can find polynomials f_j of degree 1 such that $Z =$

$Z(f_1, \dots, f_{n-d})$; since W is an irreducible component of $Z(g_1, \dots, g_{n-d})$ where $g_j = f_j \circ \pi$, its dimension is at least d. □

LEMMA 8.4. *Let X be an irreducible projective variety. Then $C(X)$ the cone over X is also irreducible, and has dimension equal to* $\dim X + 1$.

PROOF. □

COROLLARY 8.5. *Let X and Y be closed irreducible subvarieties of $\mathbb{P}^n$. Assume that $d := \dim X + \dim Y - n$ is ≥ 0. Then $X \cap Y$ is nonempty and each of its irreducible components has dimension at least d.*

PROOF. Use the affine version of this applied to $C(X) \cap C(Y)$, and note that the latter is always nonempty since it contains the origin. □

2. Local dimension, dimension and fibers

DEFINITION 8.6. Let X be an algebraic variety, $x \in X$ a point. Define $\dim_x X$, the dimension of X at x, to be the maximal length of a chain of irreducible closed subsets X_i of X with $X_0 = \{x\}$. Clearly $\dim_x X \leq \dim X$.

PROPOSITION 8.7. *If X is irreducible, then $\dim_x X = \dim X$ for every $x \in X$.*

PROOF. We may assume that X is affine. Let $\pi : X \to \mathbb{A}^n$ be a finite surjective map. Let $q = \pi(x)$ and choose a sequence $\{q\} = L_0 \subset L_1 \subset \dots L_n = \mathbb{A}^n$ of affine subspaces. Define by induction $M_0 = \{x\}$, M_{i+1} an irreducible component of $\pi^{-1}(L_{i+1})$ containing M_i. It is enough to prove that $M_i \neq M_{i+1}$. Since $\pi(M_i) \subset L_i$, we have that $\dim M_i \leq i$. On the other hand by (ref) $\dim M_i \geq i$, so $\dim M_i = i$ and we are done. □

LEMMA 8.8. *Let $f : X \to Y$ be a morphism of affine varieties. Then there exist: a morphism $g : \bar{X} \to \bar{Y}$ of projective algebraic varieties, open dense embeddings $i : X \to \bar{X}$ and $j : Y \to \bar{Y}$ such that $g \circ i = j \circ f$.*

PROOF. Assume that X is closed in $\mathbb{A}^m$ and Y in $\mathbb{A}^n$. Let $\bar{Y}$ be the closure of Y in $\mathbb{P}^n$. Let X' be the closure of the graph of f in $\mathbb{A}^m \times \bar{Y}$ and $\bar{X}$ the closure of X' in $\mathbb{P}^m \times Y$. □

THEOREM 8.9. *Let $f : X \to Y$ be a surjective morphism of irreducible varieties, $r = \dim X - \dim Y$. Then*

(1) *let $y \in Y$, and F an irreducible component of $X_y = f^{-1}(y)$. Then* $\dim F \geq r$;
(2) *there exists $U \subset Y$ nonempty open and such that for $y \in U$ one has* $\dim X_y = r$.

PROOF. (i) We may assume that X and Y are affine. We may assume that $Y = \mathbb{A}^n$; otherwise, compose with a finite surjective map $\pi : Y \to \mathbb{A}^n$. Now it is enough to apply (ref) since every point is the zero locus of n functions.
(ii) We prove this by induction on r, the case $r = -1$ being empty. We can assume that X and Y are projective[1]. Assume that X is closed in $\mathbb{P}^N$, and let H be a hyperplane not containing X (it exists since the intersection of

[1]why? Use the Lemma just proven!

all hyperplanes is empty). Let $Z = H \cap X$; Z has pure dimension $\dim X - 1$. Let $g : Z \to Y$ be the restriction of f, and let $Z_1, \dots, z_m$ be the irreducible components of Z. Let $Y_i = g(Z_i)$; it is closed in Y, since X is complete and Y is separated. Let W be the union of all the Y_i's which are different from Y, and $V = Y \setminus W$. By induction there is an open subset U in W such that if $y \in U$, $X_y \cap H$ has pure dimension $r - 1$ (which implies empty if $r = 0$). Since X_y is nonempty, $X_y \cap H$ is also nonempty unless $\dim X_y = 0$. In any case, we can deduce $\dim X_y \leq r$ and we are done. □

Add examples.

CHAPTER 9

Tangents and differentials

1. The local ring at a point

DEFINITION 9.1. Let X be an algebraic variety, $x \in X$ a point. We call *local ring* of X at x and denote by $\mathcal{O}_{X,x}$ the ring of germs of regular functions at x; that is, the quotient of the set of pairs (U, f), where U is an open neighborhood of x and $f \in \mathcal{O}_X(U)$, with respect to the relation generated by $(U, f) \equiv (V, f|_V)$ when $V \subset U$ is another open neighborhood of x.

LEMMA 9.2. *The ring $\mathcal{O}_{X,x}$ is local, with maximal ideal $\mathfrak{m}_x$ given by germs of functions vanishing at x and residue field K. If X is affine, $\mathcal{O}_{X,x}$ is the localization of $K[X]$ at the maximal ideal n_x of functions vanishing at x.*

PROPOSITION 9.3. *If $f : X \to Y$ is a morphism of algebraic varieties, and $x \in X$, then f induces a functorial homomorphism $f^* : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. If f is a closed embedding, then f^* is surjective, and if it is an open embedding, then it is an isomorphism.*

REMARK 9.4. Since it is the localization of a Noetherian ring, $\mathcal{O}_{X,x}$ is also Noetherian. It is however in general not finitely generated over K as an algebra.

LEMMA 9.5. *The intersection of all powers of $\mathfrak{m}_x$ is zero.*

PROOF. Let $M = \bigcap \mathfrak{m}_x^i$. M is an ideal in $\mathcal{O}_{X,x}$, hence it is finitely generated as a module. Clearly $M = \mathfrak{m}M$; therefore by Nakayama's lemma, $M = 0$. □

Add: X is irreducible at x (explain) iff $\mathcal{O}_{X,x}$ is a domain.

2. Cotangent space

DEFINITION 9.6. Let X be an algebraic variety, $x \in X$. We call *cotangent space* of X at x and denote by $\Omega_{X,x}$ the quotient space $\mathfrak{m}_x/\mathfrak{m}_x^2$. If $f \in \mathcal{O}_{X,x}$, we define $df \in \Omega_{X,x}$ to be the class of $f - f(x)$.

REMARK 9.7. By Nakayama, the dimension of the cotangent space equals the minimal number of generators of the maximal ideal of x in $\mathcal{O}_{X,x}$. In particular, by Noetherianity $\Omega_{X,x}$ is a finite dimensional vector space.

DEFINITION 9.8. Let X be an algebraic variety, $x \in X$, V a K-vector space. A linear map $v : \mathcal{O}_{X,x} \to V$ is called a *derivation* if

$$v(fg) = f(x)v(g) + g(x)v(g).$$

PROPOSITION 9.9. *The map $d : \mathcal{O}_{X,x} \to \Omega_{X,x}$ is a derivation.*

PROPOSITION 9.10. *Let X be an affine variety, $x \in X$, and $n_x \subset K[X]$ the maximal ideal it defines. Then the natural map $n_x/n_x^2 \to \Omega_{X,x}$ is an isomorphism.*

COROLLARY 9.11. *For any point $p \in \mathbb{A}^n$ the elements $dx_1, \dots, dx_n$ are a basis of Ω_x.*

COROLLARY 9.12. *The dimension of $\Omega_{X,x}$ is upper semicontinuous; that is, the locus of points where the dimension is fixed is locally closed, and its closure contains only points where the dimension is higher.*

DEFINITION 9.13. If $f : X \to Y$ is a morphism of algebraic varieties, and $x \in X$, $y = f(x)$, we define alinear map $f^* : \Omega_{Y,y} \to \Omega_{X,x}$ by $f^*(dg) = d(f^*g)$.

3. Tangent space

DEFINITION 9.14. The *tangent space* of X at x is the dual vector space to $\Omega_{X,x}$. For any morphism $\phi : X \to Y$ we denote by $d\phi : T_{X,x} \to T_{Y,\phi(x)}$ the dual of the map ϕ^* among the cotangent spaces and call it *differential* of f.

LEMMA 9.15. *The map $u \mapsto u \circ d$ induces an isomorphism of K-vector spaces between $T_{X,x}$ and the space of derivations from $\mathcal{O}_{X,x}$ to K.*

LEMMA 9.16. *If ϕ is a closed embedding, then $d\phi$ is injective; if it is an open embedding, it is an isomorphism.*

LEMMA 9.17. *If $F : \mathbb{A}^n \to \mathbb{A}^m$ is a morphism, its differential at a point p in the basis dx_i is given by the matrix of partial derivatives $\partial F_i/\partial x_j$.*

THEOREM 9.18. *Let X be a closed subset of $\mathbb{A}^n$, and $f_1, \dots, f_r$ a set of generators of $E(X)$. Let $f : \mathbb{A}^n \to \mathbb{A}^r$ be the morphism defined by the f_i. Then for every $x \in X$ one has $T_xX := \ker df(x) : T_x\mathbb{A}^n \to T_0\mathbb{A}^m$.*

PROOF. POP. □

4. Tangent cone

DEFINITION 9.19. The *tangent cone*C_xX is the closed subvariety of $T_{X,x}$ defined by the associated graded ring to $\mathcal{O}_{X,x}$.

PROPOSITION 9.20. *Let X be a closed subvariety of $\mathbb{A}^n$, and $x \in X$. Then the projectivization of the tangent cone of X at x is naturally isomorphic to the exceptional divisor of the blowup of X in x.*

PROOF. POP. We start by making precise this statement. The tangent cone is naturally contained in T_xX, which in turn is contained in $T_x\mathbb{A}^n$; the projectiveization of the latter is naturally isomorphic to the exceptional divisor E of $B = Bl_x\mathbb{A}^n$. The claim is that $\mathbb{P}(C_xX) = E \cap \hat{X}$, where $\hat{X}$ is the closure of $\varepsilon^{-1}(X) \setminus E$ in B. Assume without loss of generality that x is the origin.

Since we have to prove the equality of two subvarieties of B, it is enough to check this on each open set $U_i = D(y_i)$; assume for notational simplicity that we look at U_1. Then $U = U_1$ is $\mathbb{A}^n$ with coordinates $x, u_2, \dots, u_n$ and

$\varepsilon(x, u) = (x, xu)$; E is $Z(x)$. ε induces an isomorphism of $V = D(x) = U \setminus E$ with $W = D(x_1) \subset \mathbb{A}^n$.
Therefore the equations of $E \cap \hat{X}$ can be obtained as follows: take any equation $f \in E(X)$; compute $f(x, xu)$ and divide by x^d, where d is the nonzero homogeneous component of f of smallest degree. When you now substitute $x = 0$, all other homogeneous components vanish, and you are left with $f_d(1, u)$. But this is precisely the tangent cone.[1] □

COROLLARY 9.21. *The dimension of* $T_{X,x}$ *is greater then or equal to the dimension of* X *at* x. *If equality holds, then* X *is irreducible at* x.

PROOF. Use the notation of the proposition. Since E is locally given by one equation in B, and by definition $\hat{X} \cap E$ contains no irreducible component of $\hat{X}$, the dimension of $\mathbb{P}(C_x X)$ is equal to $\dim X - 1$. Hence $\dim C_x X = \dim X$ and the inequality follows from $C_x X \subset T_x X$.
If equality holds, we must have $C_x X = T_x X$ since $T_x X$ is irreducible. Therefore $\oplus m_x^i / m_x^{i+1}$ is isomorphic to $\mathrm{Sym}^* \, \Omega_x X$, which is a domain. We have to prove that $\mathcal{O}_{X,x}$ is a domain. Let $f, g \in \mathcal{O}_{X,x}$ be nonzero elements. Then there exist r, s such that $f \in \mathfrak{m}_x^r \setminus \mathfrak{m}_x^{r+1}$ and $g \in \mathfrak{m}_x^s \setminus \mathfrak{m}_x^{s+1}$. So $[f] \in \mathfrak{m}_x^r / \mathfrak{m}_x^{r+1}$ is nonzero, and same for $[g]$. But then $[f][g] = [fg]$ is nonzero, therefore fg must also be nonzero. □

COROLLARY 9.22. *A smooth variety is irreducible if and only if it is connected.*

COROLLARY 9.23. *A smooth hypersurface* X *in* $\mathbb{P}^n$ *is irreducible if* $n \geq 2$.

PROOF. The irreducible components X_i of X are hypersurfaces, hence any two of them intersect because $\dim X_i + \dim X_j - n = (n-1) + (n-1) - n \geq 0$. Therefore X cannot be smooth at any point where two different components meet. □

EXERCISE 9.24. Show that there are smooth reducible hypersurfaces in $\mathbb{A}^n$ for any n. Can you find an example (say with $n = 2$) where not all irreducible components are lines? Where no irreducible component is a line?

EXERCISE 9.25. Prove that if X is a closed subvariety of $\mathbb{A}^n$ whixh is smooth at x, then $\hat{X}$ is smooth at every point of $\hat{X} \cap E \subset Bl_x \mathbb{A}^n$.

5. Smoothness

DEFINITION 9.26. An algebraic variety X is *smooth* at x if $\dim T_{X,x} = \dim_x X$. We say that X is *smooth* if it is smooth at every point.

Note that if $U \subset X$ is an open subset, then for $x \in U$ U is smooth at x iff X is smooth at x. Moreover, $\mathbb{A}^n$ is smooth; hence $\mathbb{P}^n$ is also smooth.

PROPOSITION 9.27. *Let* X *be an algebraic variety. Then the locus of points where* X *is smooth is open.*

PROOF. The statement is local, so we may assume that X is affine and indeed closed in $\mathbb{A}^n$. We may also assume that X is irreducible; in fact, the locus of points in X lying in a unique irreducible component is open

[1] OK, questa non è scritta bene. Ma meglio che niente. Potete leggerla sul Kempf.

and dense. Let $d = \dim X$, and let $f_1, \dots, f_r$ be generators of $E(X) \subset K[x_1, \dots, x_n]$. Then $x \in X$ is smooth if and only if $df(x)$ has maximal rank, that is $n - d$. This is given by nonvanishing of the determinant of at least one $(n-d) \times (n-d)$ minor, hence it is open. □

In general, the above proof shows that the locus of points in X where $\dim T_x X$ is less than or equal to a given integer is open.

THEOREM 9.28. *Let X be an algebraic variety. Then the locus of smooth points is dense in X.*

PROOF. Again, we may assume X irreducible and closed in $\mathbb{A}^n$, with $n = d + r$ and $d \dim X$. If $r = 0$ then $X = \mathbb{A}^n$ and we are done. So we proceed by induction on r. Let $f \in E(X)$ be an element of minimal degree. We claim that there is $i \in \{1, \dots, n\}$ and $p \in X$ such that $\partial f / \partial x_i(p) \neq 0$.
The claim implies the induction step: let Y be the hypersurface $Z(f)$. Can find coordinates such that $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$ induces a finite map $f : Y \to \mathbb{A}^{n-1}$ and such that $d\pi(p) : T_p Y \to T_\pi(p)\mathbb{A}^n$ is an isomorphism. Let $Z = \pi(X)$. Then Z has a smooth open dense subset V by induction, however the locus W in Y where $d\pi$ is an isomorphism is also open, hence it is enough to take $U = W \cap \pi^{-1}(V)$.
Proof of the claim. If not, then $\partial f / \partial x_i \in E(X)$ for all i. Since its degree is smaller then $\deg f$, it must be zero. In characteristic zero this implies that f is a constant in which case X is empty, and we are done. In characteristic p, it implies that f is a p-th power, $f = g^p$. But since $E(X)$ is radical, it must be that $g \in E(X)$, and $\deg g < \deg f$, against assumptions. □

THEOREM 9.29. *Let X be an algebraic variety, and assume that the group of automorphisms of X acts transitively (i.e., given two points $x, y \in X$ there exists an isomorphisam $\phi : X \to X$ such that $\phi(x) = y$. Then X is smooth.*

PROOF. Let $x \in X$ be a smooth point (which exists since the locus of smooth points is nonempty). Then for any other $y \in X$, the isomorphism ϕ in the definition guarantees that X is also smooth at y. □

DEFINITION 9.30. An *algebraic group* is an algebraic variety G which is also a group and such that the group structure maps multiplication $m : G \times G \to G$ and inverse $i : G \to G$ are morphisms.

EXAMPLE 9.31. The following subsets of the $n \times n$ matrices with K coefficients are algebraic groups: $GL(n, K)$ (those with nonzero determinant); $SL(n, K)$ (those with determinant one); $O(n, K)$ (those satisfying $AA^t = I$); $SO(n, K) := O(n, K) \cap SL(n, K)$.

COROLLARY 9.32. *An algebraic group is smooth.*

EXERCISE 9.33. Prove that an algebraic group is separated.

6. Applications

THEOREM 9.34. *(Bertini) Let X be a smooth closed subvariety of $\mathbb{P}^n$, of pure dimension d. Then there is a nonempty open subset U of the dual projective space $\check{\mathbb{P}}^n$ such that for $H \in U$ the variety $X \cap H$ is smooth of pure dimension $d - 1$.*

PROOF. We may assume X is irreducible. Let us consider inside $\mathbb{P}^n \times \check{\mathbb{P}}^n$ the subset Γ_X of pairs (x, H) such that $x \in X \cap H$ and $T_x X \subseteq T_x H$.
Γ_X is a closed subvariety, since it is the intersection of the zero loci of f, $\sum a_i x_i$,

$$\text{rank}\begin{pmatrix} a_0 & a_1 & \dots & a_n \\ f_0(x) & f_1(x) & \dots f_n(x) & \end{pmatrix} \leq 1$$

where f varies over a set of homogeneous generators of $E(X) \subset K[x_0, \dots, x_n]$ and f_i denots $\partial f / \partial x_i$.
The fiber of Γ_X over each point of X is isomorphic to $\mathbb{P}^{n-d-1}$; hence the dimension of Γ_X is $n-1$ and its image in $\check{\mathbb{P}}^n$ is a proper closed subset. Its complement is the required open subset U. □

COROLLARY 9.35. *The general hypersurface of degree d in $\mathbb{P}^n$ is smooth, hence (if $n \geq 2$) irreducible.*

PROOF. Hypersurfaces of degree d in $\mathbb{P}^n$ are the same as hyperplane sections of the degree d Veronese embedding. Or, one can give a direct proof in this case. □

In fact, a very useful tool is a generalization of Bertini's theorem, the Bertini-Sard theorem, which claims that if $f : X \to Y$ is a morphism of smooth varities, then there is a dense open subset in U such that the fibers are smooth and the morphism has surjective differential at every point of $F^{-1}(U)$. However, this result is only true if the ground field has characteristic zero. If one is only interested in characteristic zero, one can introduce differentials at an early stage and do the whole theory of dimension in terms of dimension of tangent spaces.

PROPOSITION 9.36. *Let C be a curve, i.e. an irreducible variety of dimension 1, $p \in C$. The following are equivalent:*

(1) *C is smooth at p;*
(2) *$\mathcal{O}_{C,p}$ is a discrete valuation ring [see the algebra appendix];*
(3) *$\mathcal{O}_{C,p}$ is integrally closed.*

PROOF. The first and second condition are equivalent by Nakayama, since a local domain is a discrete valuation ring iff the maximal ideal is principal. Every discrete valuation ring is integrally closed (see appendix). □

CHAPTER 10

Rational functions and maps

1. Rational functions and function fields

DEFINITION 10.1. Let X be an irreducible algebraic variety. A *rational function* is an equivalence class of pairs (f, U) with $U \subset X$ open dense and $f \in \mathcal{O}_X(U)$; here $(f, U) \circ (g, V)$ if and only if $f|_{U \cap V} = g|_{U \cap V}$.

EXERCISE 10.2. This is an equivalence relation. Hint: you need to view regular functions as morphisms to $\mathbb{A}^1$ and use the fact that $\mathbb{A}^1$ is separated.

LEMMA 10.3. *Rational functions on X are a field. If X is affine, they are the quotient field of the coordinate ring $K[X]$.*

DEFINITION 10.4. The field of rational functions on X is called the *function field* of X and denoted by $K(X)$.

LEMMA 10.5. *Let X be an irreducible algebraic variety, $(f, U) \in K(X)$ then there is a (unique) open subset V in X such that there is a pair $(V, g) \equiv (U, f)$ and for every other equivalent pair (W, h) one has $W \subset V$. V is called* domain of definition *of (U, f).*

LEMMA 10.6. *If $\phi : X \to Y$ is a dominant morphism of irreducible algebraic varieties, then it induces a field homomorphism $\phi^* : K(Y) \to K(X)$. If ϕ is the inclusion of an open subset, then ϕ^* is an isomorphism.*

2. Rational maps

DEFINITION 10.7. Let X and Y be varieties. A *rational map* from X to Y is an equivalence class of pairs (f, U) where U in X is open and dense and $f : U \to Y$ is a morphism. (f, U) is equivalent to (g, V) if $f|_{U \cap V} = g|_{U \cap V}$. Note that $U \cap V$ is nonempty and dense; moreover, if $(f, U) \equiv (g, V)$ then there exists $(h, U \cup V) \equiv (f, V)$.

LEMMA 10.8. *Every rational map has a maximal open subset where it is a morphism, called its domain of definition.*

DEFINITION 10.9. Let $F : X - \to Y$ and $G : Y - \to Z$ be rational maps. Then the *composition* $G \circ F$ is defined as a rational map from X to Z iff the inverse image in X of the domain of definition of G is nonempty. In particular it is always defined if X is irreducible and F is dominant.

COROLLARY 10.10. *Irreducible separated varieties and dominant morphisms form a category*

PROPOSITION 10.11. *Let X and Y be irreducible separated projective varieties. To give a dominant rational map $X - \to Y$ is the same as to give an embedding of fields $K(Y) \to K(X)$.*

PROOF. We may assume that X and Y are affine. Given an inclusion $j : K(Y) \to K(X)$, let $y_1, \dots, y_r$ be generators of $K[Y]$ as a K algebra. Write $f_i := j(y_i)$. Let U_i in X be the domain of definition of f_i, and U a nonempty affine open subset contained in the intersection of the U_i's. Then j induces an injective homomorphism $K[Y] \to K[U]$, hence a dominant morphism $U \to Y$. □

LEMMA 10.12. *If C is a smooth curve, then every rational map from C to $\mathbb{P}^N$ is a morphism.*

PROOF. Let $f : C - \to \mathbb{P}^n$ be a rational map, V its domain of definition. So V is the complement of a finite set in C. Let c belong to the complement. We want to prove that f is indeed regular also in c. Choose coordinates such that the inverse image W of $U_0 := \{x_0 \neq 0\}$ in U is nonempty. Then, f is induced by a morphism $W \to U_0$, that is by an $(n+1)$-tuple $(1, f_1, \dots, f_n)$ where $f_i \in K[W] \subset \mathcal{O}_{C,c}$. Let $d_i = o_p(f_i)$, and $d_0 = 0$. Let $d := -(\min d_i) \geq 0$, and t a local coordinate for C near c. Then $(t^d, t^d f_1, \dots, t^d f_n)$ defines the same map on W but is also regular on c, since all functions $t^d f_i$ are regular and at least one does not vanish. □

COROLLARY 10.13. *Every rational map from a smooth curve to a projective variety is a morphism.*

3. Birational maps

DEFINITION 10.14. A *birational map* is a dominant rational map $f : X - \to Y$ such that there exists a (necessarily unique) dominant rational map $g : Y - \to X$ with $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. If such f and g exist, we say that X and Y are birational.

REMARK 10.15. Isomorphic varieties are birational. If X is irreducible and separated and U is a nonempty open subset, then X is birational to U.

COROLLARY 10.16. *Two varieties are birational if and only if their function fields are isomorphic.*

PROPOSITION 10.17. *Two varieties X and Y are birational, if and only if they contain isomorphic nonempty open subsets.*

PROOF. One implication is already included by the remark above. Conversely, assume that $f : X - \to Y$ is defined by $f : U \to Y$, and that U and Y are affine. Then f induces an inclusion $K[Y] \to K[U]$ which induces an isomorphism on function fields. Let $f_1, \dots, f_r$ be generators of $K[U]$ as $K[Y]$ algebra. Then there exist $a_i, b_i \in K[Y]$ such that $f_i = a_i/b_i$. Let $b \in K[Y]$ be the product of the b_i. Then the inclusion $K[U]_h \to K[Y]_h$ is in fact an isomorphism, since each f_i is in the image. Therefore f induces an isomorphism $f^{-1}(V) \to V$, where $V = D(h) \subset Y$. □

REMARK 10.18. Every irreducible variety is birational to an affine and to a projective variety.

More is true: every irreducible variety is birational to a hypersurface. We only sketch a proof.

COROLLARY 10.19. *Every birational map between smooth projective curves is an isomorphism.*

CHAPTER 11

Hilbert polynomial

Note: this is not part of the program this year.

Let $R = K[x_0, \dots, x_n]$ and $S = R/J$ where J is a homogeneous ideal (i.e., one generated by homogeneous polynomials). If J is radical and not equal to $(x_0, \dots, x_n)$ then it is the ideal of a projective variety in $\mathbb{P}^n$, and in general $Z(J)$ will be a subvariety. We define the Hilbert function of S as follows: $HF(S,i) := \dim S_i$ where S_i is the degree i part of S (i.e., the quotient of the vector space of degree i polynomials on S by the subspace of degree i polynomials in J). We extend HF to a function from the integers to the integers by declaring $HF(i) = 0$ for all $i < 0$.

EXAMPLE 11.1. The Hilbert function of R is $\binom{n+t}{n}$.

LEMMA 11.2. *Let $f \in S_i$ be an element which is not a zero divisor in S. Let $S' = S/(f) = R/(J + (f))$. Then*

$$HF(S', m) = HF(S, m) - HF(S, m - i).$$

PROOF. There is an exact sequence

$$0 \to S_{m-i} \to S_m \to S'_m \to 0$$

where the first map is multiplication by f and the second map is the natural projection. □

EXAMPLE 11.3. Let $J = (f)$ with f a homogeneous polynomial of degree d. Then $HF(S,m) = \binom{m+n}{n}$ for $m < d$ and to $\binom{m+n}{n} - \binom{m+n-d}{n-d}$ otherwise. In particular for $m \geq d$ it is a polynomial of degree $n-1$ and leading coefficient $\frac{d}{(n-1)!}$.

Let $F : \mathbb{Z} \to \mathbb{Z}$ be any function. We define its difference function ΔF by $\Delta F(m) := F(m) - F(m-1)$. We say that F is eventually polynomial if there exists a polynomial function f such that $F(m) = f(m)$ for all $m >> 0$. f is called the associated polynomial of F.

LEMMA 11.4.

(1) *The associated polynomial is unique: it is a combination with integral coefficients of the polynomials $\binom{t+j}{j}$.*
(2) *The polynomial $\binom{t+j}{j}$ has degree j and leading coefficient $\frac{t^j}{j!}$.*
(3) *If ΔF is eventually polynomial with associated polynomial of degree d, then F is eventually polynomial with associated polynomial of degree $d+1$.*

PROOF. (1) Uniqueness follows from the fact that, as $\mathbb{Z}$ is infinite, two different polynomials can have the same value only at a finite number of points. It is immediate that the functions $\binom{t+j}{j}$ are integer valued and have the given degree and leading coefficient. It is then easy to prove by induction on the degree that any polynomial function that is eventually integer valued must be a linear combination of them with integral coefficients. I'll fix this later. □

THEOREM 11.5. *$HF(S)$ is eventually polynomial and the associated polynomial has degree equal to the dimension of $Z(J)$ (and is zero iff $Z(J)$ is empty).*

PROOF. Induction on $\dim Z(J)$. Take a linear function $g \in R_1$ which does not vanish on any component of $Z(J)$. Etc. □

LEMMA 11.6. *Let J be a homogeneous ideal in $K[x_0, \ldots, x_n]$ with generators $f_1, \ldots, f_m$ and assume that at every point p of $Z(J)$ the matrix $\partial f_i/\partial x_j$ has rank $n - \dim Z(J)$. Then J is a radical ideal (and, of course, $Z(J)$ is smooth).*

PROOF. boh. □

EXAMPLE 11.7. Let X be an irreducible projective variety of dimension d in $\mathbb{P}^n$. Then the leading coefficient of the Hilbert polynomial of X is $Dt^d/d!$ where D is the number of points of intersection of X with a general linear space of dimension $n - d$.

THEOREM 11.8. *Bézout's theorem for plane curves.*

THEOREM 11.9. *Some general version of Bézout's theorem*

Compute the Hilbert function and the Hilbert polynomial of 2 distinct points, of 3 distinct points. Compute the Hilbert polynomial of N distinct points for any N. Define the Hilbert scheme. Compute the Hilbert scheme of the projective line. Show that the moduli of hypersurfaces and the Grassmannian are indeed components of the Hilbert scheme corresponding to a given polynomial. Work out the example in Hartshorne (projecting the normal cubic curve from a point). Mention that the Hilbert scheme can be defined locally by the algebraic condition of flatness and the two are connected by 1) flat +fg is locally free 2) Serre vanishing.

CHAPTER 12

Final exercises

EXERCISE 12.1. Fix integers $0 < r < n$. Let $\Gamma \subset \mathbb{P}^n \times G(r+1, n+1)$ be the incidence variety, that is the set of pairs (x, Λ) where Λ is a linear projective subspace of $\mathbb{P}^n$ of dimension r and $x \in \Lambda$. Let $p : \Gamma \to \mathbb{P}^n$ and $q : \Gamma \to G := G(r+1, n+1)$ be the projections.
Prove that there exists a variety F and an open cover U_i of $\mathbb{P}^n$ such that $p^{-1}(U_i)$ is isomorphic to $U_i \times F$ compatibly with the projection to U_i. Prove that Γ is not isomorphic to $F \times \mathbb{P}^n$ compatibly with the projection p. We say that p is locally, but not globally, a product. Prove that also q is locally, but not globally, a product.

EXERCISE 12.2. Prove that if $\phi : U \to \mathbb{P}^m$ is a morphism, and $U \subset \mathbb{P}^n$ is a nonempty open subset, then there exists $d > 0$ and homogeneous polynomials $f_0, \dots, f_m \in K[x_0, \dots, x_n]$ of degree d with no common factor such that $U \cap Z(f_0, \dots, f_m) = \emptyset$ and $\phi(p) = (f_0(p), \dots, f_n(p))$ for every $p \in U$. Such a d is called the degree of ϕ.
Prove that the f_i are unique up to multiplication by a nonzero scalar.
Prove that the automorphism group of $\mathbb{P}^n$ is $PGL(n+1, K)$, that is the quotient of $GL(n+1, K)$ by the scalar multiples of the identity.

EXERCISE 12.3. Prove that $\mathbb{P}^1$ is not isomorphic to a smooth plane curve C of degree 3. That is, there exist nonrational varieties! To do this, you must use the result of the previous exercise. The proof goes by assuming that such an isomorphism exists and deducing a contradiction.

(1) Prove that if $\phi : \mathbb{P}^1 \to \mathbb{P}^2$ induces an isomorphism $\mathbb{P}^1 \to C$, then its degree is 3.
(2) Prove that there is a linear projection from D, the Veronese embedding of degree 3 of $\mathbb{P}^1$, to C, which is an isomorphism.
(3) Prove that every point in $\mathbb{P}^3$ lies in a line which is either tangent to D or intersects D in ≥ 2 points.
(4) Complete the proof by using the fact that an isomorphism is injective and injective on tangent spaces.

EXERCISE 12.4. Blowup of a variety at a point. If X is a closed subvariety of $\mathbb{A}^n$ containing zero, define Bl_0X to be the closure in the blowup of $\mathbb{A}^n$ of $\varepsilon^{-1}(X \setminus 0\}$, and let $\varepsilon_X : Bl_0X \to X$ be the restriction of ε. Prove that if $U \subset X$ is an open subset containing 0 and $\phi : U \to Y$ is an isomorphism, where Y is a closed subvariety of $\mathbb{A}^m$ and $\phi(0) = 0$, then there is a unique isomorphism $\psi : \varepsilon_X^{-1}(U) \to Bl_0Y$ such that $\varepsilon_Y \circ \psi = \phi \circ \varepsilon_X|_{\varepsilon_X^{-1}(U)}$.
Deduce that we can define the blowup of any algebraic variety at a point (by taking the blowup on an affine open set and gluing the rest of the variety unchanged.

Prove that $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in a poit is isomorphic to $\mathbb{P}^2$ blown up in 2 points.

Exercise 12.5. Let X be an algebraic variety, and G its automorphism group. We say that G acts transitively if for every $x, y \in X$ there exists $g \in G$ such that $g(x) = y$. In this case we also say that X is homogeneous. Prove that a homogeneous algebraic variety is smooth. Prove that $\mathbb{P}^n$ and $G(r, n)$ are homogeneous.
Let $X = Bl_0\mathbb{A}^2$. Prove that X contains only one projective curve, namely the exceptional divisor. Deduce that X is not homogeneous.

Exercise 12.6. Let $X = \mathbb{P}^n$, and assume we are given triplets of distinct points a, b, c and x, y, z in X (i.e., no two of $\{a, b, c\}$ are equal and no to of $\{x, y, z\}$ are equal). Prove that there is an automorphism ρ of X such that $\rho(a) = x$, $\rho(b) = y$.
Prove that if $n = 1$ we can choose ρ such that morever $\rho(c) = z$ and in this case ρ is unique. What happens in this case of $n > 1$?

Exercise 12.7. Let $G := G(2, 4)$ be the Grassmann variety parametrizing lines in $\mathbb{P}^3$. The Plücker embedding realizes G as a quadric in $\mathbb{P}^5$. Prove that the set of lines meeting a given line is a hyperplane section of G in the Plücker embedding. Discuss which hyperplane sections can be obtained in this way. Prove that G contains two kinds of projective linear subspaces of dimension 2; the lines containing a given point and the lines contained in a given plane.

Exercise 12.8. If X is a separated algebraic variety, the intersection of two open affines is affine. Find an example where this is not true.

APPENDIX A

Commutative algebra

1. Basic concepts

This is a very short summary of the commutative algebra definitions and facts which will be needed to properly understand the course.

1.1. Rings. A *ring* will be a set with two operations, sum and product, satisfying the usual commutative, associative and distributive property we know from the integers; we will moreover assume that there are neutral elements for addition (denoted 0) and multiplication (denoted 1) and that $0 \neq 1$. We will also assume that every element a has an additive inverse $-a$; if it has a multiplicative inverse, it's called a *unit*. If every nonzero element is a unit, the ring is called a *field*. If a and b are nonzero such that $ab = 0$, then a and b are called *zero divisors*. If there are no zero divisors, then the ring is called *domain*. Every field is a domain. An element $a \in A$ is nilpotent if there is n such that $a^n = a \cdot \ldots \cdot a = 0$. A ring is reduced if it has no nilpotents. Prove that field implies domain implies reduced.

A *(homo)morphism* of rings is a map respecting the two operations and 1. Rings are a category (i.e., identity is a homomorphism and the composition of homomorphisms is a homomorphism).

1.2. Modules. A *module* over a ring A is an abelian group M together with a bilinear multiplication $A \times M \to M$ satisfying $1 \cdot m = m$ and $(a_1 a_2)m = a_1(a_2 m)$ for every $a_1, a_2 \in A$ and $m \in M$. A (homo)morphism of A-modules $M \to N$ is an A-linear group homomorphism. If A is a field K, then an A-module is a K-vector space.

A^n has a natural structure of A-module for every $n > 0$. It enjoys the following universal property: to give a homomorphism $A^n \to M$ is equivalent to giving the images of the n elements

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1).$$

An A-module M is called *finitely generated* if there exists a surjective homomorphism $A^n \to M$ for some $n > 0$. A f.g. A-module M which is isomorphic to A^n for some $n > 0$ is called free; if A is not a field, not every f.g. A-module is free. In particular for a general f.g. module there is no obvious analog of the dimension of a vector space, or of a basis.

1.3. Ideals. A sub-A-module of A is called an *ideal*. A subset I of A is an ideal if and only if it is a subgroup with respect to addition and is closed under multiplication by arbitrary elements of A. A proper subgroup I of A is an ideal if and only if the multiplication of A induces a well-defined multiplication and thus a ring structure on the quotient. The ideal (S)

generated by a subset S of A is the smallest ideal containing S; it is the set of finite A-linear combinations of elements of S.

An ideal I is called maximal the only ideal which properly contains it is the whole ring. It is called prime if $ab \in I$ implies either $a \in I$ or $b \in I$ (or both!). It is called radical if $\sqrt{I} := \{a \in A \mid \exists n > 0 \text{s. t. } a^n \in I\}$ is equal to I. Prove that maximal implies prime implies radical. Prove that an ideal I in a ring A is maximal iff A/I is a field, it is prime iff A/I is a domain, and is radical if A/I has no nilpotents (a nilpotent being a nonzero element b such that $b^n = 0$ for some $n > 1$). Prove that for any ideal I the subset $\sqrt{I}$ is always an ideal. Prove that $\sqrt{0}$ is the set of nilpotent elements (it's called the nilradical). Prove that in general the set of zero divisors is not an ideal. Hint: Look at the ring $K[x]/(x^2 + x)$.

1.4. Algebras. If A is a ring, an A-*algebra* is a ring B together with a multiplication $A \times B \to B$ which makes B into an A-module and satisfies $(a_1 b_1)(a_2 b_2) = (a_1 a_2)(b_1 b_2)$ for $a_i \in A$ and $b_i \in B$. It is easy to see that to give a ring B an A-algebra structure is equivalent to defining a homomorphism $A \to B$. A morphism of A-algebras is a ring homomorphism which is also a module homomorphism.

The "easiest" A-algebras are the polynomials $A[x_1, \dots, x_n]$; the official definition is as formal sums $\sum_I a_I x^I$ where I ranges over all multi-indices $I = (i_1, \dots, i_n) \in \mathbb{N}^n$, $x^I = x_1^{i_1} \cdot \dots \cdot x_n^{i_n}$, the a_I's are elements of A and only a finite number of the a_I's are nonzero. Every polynomial $f \in A[x_1, \dots, x_n]$ defines a function $A^n \to A$ in the obvious way. Note that, if A is finite, then different polynomials may well define the same function.

The ring of polynomials satisfies the following universal property: given an A-algebra B, to give an A-algebra morphism $A[x_1, \dots, x_n] \to B$ is equivalent to selecting n elements of B (the images of the x_i's). If there is a surjective morphism $A[x_1, \dots, x_n] \to B$ for some n we say that B is a *finitely generated* A-algebra.

Note that $A[x_1, \dots, x_n][y]$ is isomorphic to $A[x_1, \dots, x_{n+1}]$.

2. Local rings and Nakayama's lemma

A ring A is *local* if it has a unique maximal ideal $\mathfrak{m}$. Show that A is local if and only if the elements that are not units are an ideal (necessarily $\mathfrak{m}$). The field $A/\mathfrak{m}$ is called the *residue field* of A. The name comes from the fact that the ring of germ of functions at a point ion a space with functions (or, the stalk of $\mathcal{O}_X$ at every point) is a local ring. A result you need to now **by heart** is the following, usually referred to as Nakayama's lemma.

LEMMA A.1. *Let A be a local ring, and M a finitely generated A module. If $M = \mathfrak{m}M$, then $M = 0$.*

PROOF. Let $m_1, \dots m_r$ be generators of M. Then $\mathfrak{m}M$ is generated by all elements of the form am_i, with $a \in \mathfrak{m}$. In particular for each $i = 1, \dots, r$, we can find $a_{ij} \in \mathfrak{m}$ such that $m_i = \sum_{j=1}^r a_{ij} m_j$. In other words, if $m \in M^r$ is the column vector $(m_1, \dots, m_r)^t$ and A is the matrix (a_{ij}), then $m = Am$, or $Bm = 0$ where $B = I_r - A$. Let C be the adjoint of B, hence $CB = BC = d \cdot I_r$ where $d = \det B$. Clearly $CBm = 0$, hence $dm_i = 0$ for every $i = 1, \dots, r$. It is easy to see that $d - 1 \in \mathfrak{m}$, hence d is a unit. □

COROLLARY A.2. *Let M be a finitely generated A module, $m_1, \dots, m_s \in M$. If the m_i generate $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$ vector space, then they generate M as an A module.*

3. Localization

Let A be a ring, $S \subset A$ a multiplicative set (i.e., given $s, s' \in S$, we have $ss' \in S$. Let M be an A module (for instance, A itself). Define the localization[1] $S^{-1}M$ as the following set. The elements are equivalence classes of symbols m/s, where $m \in M$ and $s \in S$, and m/s is equivalent to $\bar{m}/\bar{s}$ iff there exists $s' \in S$ such that $s'(\bar{s}m - s\bar{m}) = 0$.

Notice that $S^{-1}M$ has a sum defined by

$$\frac{m}{s} + \frac{\bar{m}}{\bar{s}} := \frac{\bar{s}m + s\bar{m}}{s\bar{s}}$$

which makes it into an abelian group. If M is an A algebra R, then $S^{-1}R$ is also a ring with multiplication

$$\frac{r}{s} \cdot \frac{\bar{r}}{\bar{s}} := \frac{r\bar{r}}{s\bar{s}}.$$

In particular, $S^{-1}A$ is an A algebra via the natural homomorphism $a \mapsto a/1$. You can prove for yourself, look up or believe a number of useful properties.

EXERCISE A.3. A ring homomorphism $\phi : A \to B$ factors via $S^{-1}A$ iff $\phi(s)$ is a unit in B for every $s \in S$; such a factorization if it exists is unique. $S^{-1}M = M \otimes_A S^{-1}A$.

LEMMA A.4. *$S^{-1}A = 0$ iff $0 \in S$.*

PROOF. We leave the proof, which is very easy, to the reader. This is however a key step in many theorems. □

The most important examples of localization are:

(1) localization at an element $f \in A$, where $S = \{1, f, f^2, f^3, \dots\}$; in this case one writes M_f for $S^{-1}M$;
(2) localization at a prime ideal $\mathfrak{p}$, where $S = A \setminus \mathfrak{p}$; in this case one writes $M_\mathfrak{p}$ for $S^{-1}M$.

EXERCISE A.5. Prove that if $f \in A$, then A_f is canonically isomorphic to $A[y]/(yf - 1)$. Prove that if $\mathfrak{p}$ is prime, then $A_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}A_\mathfrak{p}$ and residue field the fraction field of $A/\mathfrak{p}$. In particular if A is local, then $A \to A_\mathfrak{m}$ is an isomorphism.

4. Unique factorization domains

DEFINITION A.6. Let A be a domain. An element $f \in A$ is called *irreducible* if whenever $f = gh$, then either g or h is a unit. A nonzero element f is *prime* if whenever f divides gh, then it divides either g or h (in other words, the ideal (f) is prime).

[1]This looks a lot like the construction of the rationals from the integers doesn't it? And that weird extra s' you can get rid of if you assume that M has no torsion, that is $am = 0$ implies either $a = 0$ or $m = 0$. But you don't want to assume that in general.

DEFINITION A.7. A domain A is called a *unique factorization domain*, usually shortened as UFD, if

(1) every irreducible is prime, and conversely;
(2) every non zero element a in A can be written as $a = uf_1 \cdot \ldots \cdot f_r$ with u a unit and f_i irreducible elements, for some $r \geq 0$;
(3) the factorization of a is essentially unique; if $a = vg_0 \cdot \ldots \cdot g_s$ is another such decomposition, then $r = s$, and there exists a permutation σ of $\{1, \ldots, r\}$ and units u_i and w such that $g_i = u_i f_{\sigma(i)}$ and $v = uw$.

EXAMPLE A.8. It is well known from primary school that the integers are a UFD. It is trivial to prove that any field is a UFD. It si true, but too long for me to type up, that if A is a UFD then $A[x]$, and hence by induction $A[x_1, \ldots, x_n]$, is a UFD. In particular, $K[x_1, \ldots, x_n]$ is a UFD.

5. Module–finite algebras

REMARK A.9. Let $\phi : A \to B$ be a ring homomorphism. Then ϕ induces on B a structure of A-algebra, via $a \cdot b := \phi(a)b$ for $a \in A$ and $b \in B$. Conversely, if B is an A-algebra, the map $\phi : A \to B$ defined by $\phi(a) := a \cdot 1$ (for $a \in A$ and $1 \in B$) is a ring homomorphism.

If B is an A-algebra and $b \in B$, we denote by $A[b]$ the A-subalgebra of B generated (as an algebra) by b.

DEFINITION A.10. An A algebra B is called *module-finite* if it is finitely generated as an A-module.

PROPOSITION A.11. *Let C be a B-algebra, and B an A-algebra. Then C has a natural structure of A algebra; if C is module finite as B–algebra and B is module-finite as A-algebra, then C is module-finite as A-algebra.*

PROOF. The first statement is immediate, by composing the structure homomorphisms $A \to B$ and $B \to C$. For the second statement, let $\{c_i\}$ be a set of generators for C as B-module and $\{b_j\}$ be a set of generators of B as an A-module. It is easy to verify that $\{b_j \cdot c_i\}$ is a set of generators for C as an A-module. □

DEFINITION A.12. Let B be an A-algebra. We say that $b \in B$ is *integral* over A if there is a monic polynomial $f \in A[t]$ such that $f(b) = 0$; equivalently, if there exist $n > 0$ and $a_0, \ldots, a_{n-1} \in A$ such that

$$b^n + \sum_{i=0}^{n-1} a_i b^i = 0.$$

PROPOSITION A.13. *Let $b \in B$, B an A-algebra. Then the following are equivalent:*

(1) *b is integral over A ;*
(2) *there exists $n > 0$ such that $1, b, \ldots, b^n$ generate $A[b]$ as an A-module.*
(3) *$A[b]$ is module-finite as an A-algebra;*

PROOF. (1) $\Rightarrow$ (2). Immediate (use the monic polynomial). (2) $\Rightarrow$ (3) Trivial. (3) $\Rightarrow$ (1). Let $f_1, \dots, f_r \in A[t]$ be polynomials such that $f_i(b)$ generators of $A[b]$ as A-module. Let $d = \max \deg f_i$, and write $b^{d+1} = \sum a_i f_i(b)$. Then $f := t^{d+1} - \sum a_i f_i$ is a monic polynomial such that $f(b) = 0$. □

COROLLARY A.14. *If A is a Noetherian ring and A is an A-algebra, then the set of elements in B integral over A is a A-subalgebra of B.*

PROOF. Let $b_1, b_2 \in B$ integral over A. We need to prove that $b_1 b_2$ and $b_1 + b_2$ are integral over A. But the algebras $A[b_1 + b_2]$ and $A[b_1 b_2]$ are both subalgebras of $A[b_1, b_2]$ which is a finitely generated, and hence Noetherian, A-module. So they are also finitely generated A-modules. □

THEOREM A.15. *Let B be a finitely generated A algebra, with A Noetherian. Then B is module-finite if and only if every element of B is integral over A.*

PROOF. Only if: If B is module-finite, it is Noetherian, hence for every $b \in B$ the subalgebra $A[b]$ is module-finite. If: Let $b_1, \dots, b_n$ be generators of B as an A algebra. Let B_i be the subalgebra generated by $b_1, \dots, b_i$ (with $B_0 = A$). Since $B_{i+1} = B_i[b_{i+1}]$ which is integral over A, and hence over B_i, we have that each B_{i+1} is module finite over B_i, hence we are done by induction on n. □

PROPOSITION A.16. *Let A be a K algebra, and B a subalgebra of A such that A is a f.g. B-module. If K is algebraically closed, then every K-homomorphism $\phi : B \to K$ can be extended to $\psi : A \to K$.*

PROOF. Let $\mathfrak{m} = \ker \phi$. By Nakayama[2] $mA \neq A$ (here we use that A is a finitely generated B module) hence it is contained in a maximal ideal n of A. Then A/n is a field which contains K and has finite dimension as K vector space. Since K is algebraically closed, we must have $A/n = K$. □

We end with some properties of module finite algebra extensions that will be needed in the rpoof of NSS.

PROPOSITION A.17. *Let $\phi : A \to B$ be a ring homomorphism, with A Noetherian, making B into a module-finite A-algebra. Let $\mathfrak{m}$ be a maximal ideal in A. Then $\mathfrak{m} \cdot B = B$ implies $\ker \phi \not\subset \mathfrak{m}$.*

PROOF. The $A_{\mathfrak{m}}$-algebra $B_{\mathfrak{m}}$ is module finite, hence by Nakayama $B_{\mathfrak{m}}$ is zero iff $B_{\mathfrak{m}} = \bar{\mathfrak{m}} B_{\mathfrak{m}}$ where $\bar{\mathfrak{m}} = \mathfrak{m} \cdot A_{\mathfrak{m}}$ is the maximal ideal of $A_{\mathfrak{m}}$. Clearly $\mathfrak{m}B = B$ implies, after localizing, that $B_{\mathfrak{m}} = \bar{\mathfrak{m}} B_{\mathfrak{m}}$, hence $B_{\mathfrak{m}} = 0$. On the other hand, it is easy to check that $B_{\mathfrak{m}} = S^{-1}B$ where $S = \phi(A \setminus \mathfrak{m})$. Therefore $B_{\mathfrak{m}} = 0$ implies $0 \in S$, hence $\ker \phi \not\subset \mathfrak{m}$. □

COROLLARY A.18. *Let A be a noetherian ring, and $\mathfrak{m}$ a maximal ideal with residue field K, an algebraically closed field. Let B be a module finite A-algebra such that the structure morphism $\phi : A \to B$ is injective. Then there exists $\psi : B \to K$ homomorphism of K-algebras such that $\ker(\psi \circ \phi) = \mathfrak{m}$.*

[2]you may notice that A is not local. However, it is enough to prove that $\mathfrak{m}A_{\mathfrak{m}} \neq A_{\mathfrak{m}}$,a dn that follows from Nakayama.

PROOF. Let $\mathfrak{n} = \mathfrak{m} \cdot B$. By the previous theorem, $\mathfrak{n}$ is a proper ideal. Hence, it is contained in a maximal ideal $\mathfrak{m}'$. Let $\psi : B \to L$ be the quotient by $\mathfrak{m}'$; L is a field which is finitely generated as a K-module. Since K is algebraically closed, we must have $K = L$. □

6. Noether's normalization lemma: the algebra side

In this section K will be a field, not necessarily algebraically closed except when this is explicitly stated. This proof may not make very much sense now, but we will go back to it later since it contains important ideas.

LEMMA A.19. *Let K be an infinite field, $f \in K[x_1, \dots, x_n]$ a nonzero polynomial. Then there exists $p \in K^n$ such that $f(p) \neq 0$.*

PROOF. Induction on n. For $n = 1$ this is clear, since a polynomial of dree d has at most d zeroes, while the field is finfinite. Write $f = \sum_{i=0}^{d} g_i x_n^i$ where $g_i \in K[x_1, \dots, x_{n-1}]$ and $g_d \neq 0$. By induction there exist $p_1, \dots, p_{n-1}$ such that $g_d(p) \neq 0$. Therefore $f(p_1, \dots, p_{n-1}, t)$ is a nonzero polynomial in t and as such cannot be identically zero by the case $n = 1$ treated before. □

LEMMA A.20. *Let K be an infinite field, and $f \in K[x_1, \dots, x_n]$ a nonzero polynomial of degree d. Then there exists a change of variables of the form $y_i = x_i + \lambda_i x_n$ (for $i < n$) and $y_n = \lambda_n x_n$ such that $g(x_i) := f(y_i)$ has the form $\sum_{i=0}^{d} a_i x_n^{d-i}$ where $a_i \in K[x_1, \dots, x_{n-1}]$ is a polynomial of degree at most i and $a_0 \in K \setminus 0$. If K is algebraically closed, we can achieve $a_0 = 1$.*

PROOF. Let f_d be the homogeneous part of degree d of f. The coefficient of x_n^d in g is $f_d(\lambda_1, \dots, \lambda_n)$. Since $f_d \neq 0$, by the previous lemma we have proven the first statement. The second half is left as an exercise. Hint: you can take d-th roots. □

LEMMA A.21. *Let K be a field, $I \subset K[x_1, \dots, x_n]$ an ideal, and assume that I contains an $f \in K[x_1, \dots, x_n]$ of the form*

$$f = x_n^d + \sum_{i=0}^{n-1} f_i(x_1, \dots, x_{n-1}) x_n^i.$$

Let $J = I \cap K[x_1, \dots, x_{n-1}]$. Then the algebra $K[x_1, \dots, x_n]/I$ is finite as a module over the subalgebra $K[x_1, \dots, x_{n-1}]/J$.

THEOREM A.22 (Noether normalization lemma). *Let K be an infinite field, A a (nonzero) finitely generated K algebra. Then there exists $r \geq 0$ and a subalgebra B of A such that: B is isomorphic to $K[x_1, \dots, x_r]$ and A is a finitely generated B module.*

PROOF. Induction on the number n of generators of A as a K algebra. If $n = 0$ there is nothing to prove. So write $A = K[x_1, \dots, x_n]/I$. If $I = 0$ take $B = A$ and $r = n$. So assume $I \neq 0$, and take $f \in I \setminus 0$. Up to a coordinate change we can assume that $f = \sum_{i=0}^{d} a_i x_n^{d-i}$ as in the previous lemma. Let A' be the image in A of $K[x_1, \dots, x_{n-1}]$. By induction A' contains a B as requested such that A' is a f.g. B-module. However, A is a f.g. A' module, and we are done. □