Ordinary (0-form) symmetries

Symmetry transf. in QFT

$$\langle U_g(\Sigma)\, \Phi^i(y)\rangle = R(g)^i{}_j \langle \Phi^j(y)\rangle$$

Since the sym. generators are CONSERVED / COMMUTE WITH HAMILTONIAN, $U_g(\Sigma)$ is "topological" (as we will see)

In Field Theory, if S is invariant under sym group G, then there exists a CONSERVED CURRENT $\partial_\mu j^\mu = 0$

j s.t. if we take local transf

$$S[\Phi^i + \epsilon(x) \underbrace{M^i{}_j}_{\text{generator}} \Phi^j] - S[\Phi^i] = -\int \epsilon(x)\, \partial_\mu j^\mu(x) \qquad (*)$$

then is a current associated with any gen.

In QFT $\rightsquigarrow$ WI

$$i\langle \partial_\mu j^\mu(x)\, \Phi^i(y)\rangle = \delta^4(x-y)\, M^i{}_j \langle \Phi^j(y)\rangle \qquad (\circ)$$

Dim. $\langle \partial_\mu j^\mu(x)\, \Phi^i(y)\rangle = N\int \mathcal{D}\Phi\, \partial_\mu j^\mu(x)\, \Phi^i(y)\, e^{iS[\Phi]} =$

$$\overset{(*)}{=} -N\int \mathcal{D}\Phi\, \frac{\delta}{\delta\epsilon(x)} S[\Phi^k + \epsilon(x) M^k{}_j \Phi^j]\Big|_{\epsilon=0} \Phi^i(y)\, e^{iS[\Phi]} =$$

$$= -\frac{1}{i}\frac{\delta}{\delta\epsilon(x)} N\int \mathcal{D}\Phi\, \Phi^i(y)\, e^{iS[\underbrace{\Phi^k + \epsilon M^k{}_j \Phi^j}_{= \Phi'^k \to \Phi^k = \Phi'^k - \epsilon M^k{}_j \Phi'^j}]}\Big|_{\epsilon=0}$$

$$= i\frac{\delta}{\delta\epsilon(x)} N\int \mathcal{D}\Phi' \left(\Phi'^i(y) - \epsilon(y) M^i{}_j \Phi'^j(y)\right) e^{iS[\Phi']}\Big|_{\epsilon=0}$$

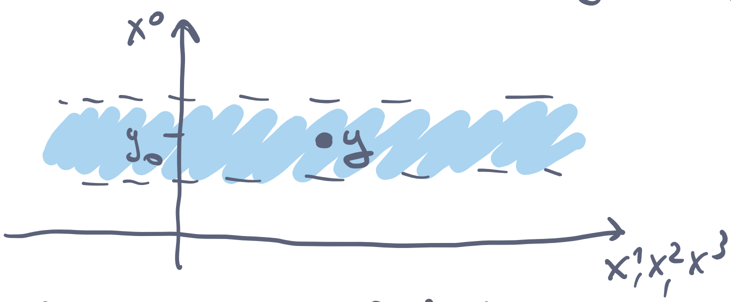
$$= -i\, \delta^4(x-y)\, M^i{}_j \langle \Phi^j(y)\rangle \quad //$$

We can now integrate the WI (o) and obtain

$$i\langle [Q, \Phi^i(y)]_{\text{eq. time}} \rangle = M^i{}_j \langle \Phi^j(y) \rangle \qquad \text{(canonical quantization)}$$

Dim. Integrate $i\langle \partial_\mu j^\mu(x)\, \Phi^i(y)\rangle = \delta^4(x-y) M^i{}_j \langle \Phi^j(y)\rangle$

over the domain $\Omega_\varepsilon \equiv [y^0+\epsilon, y^0-\epsilon] \times \mathbb{R}^3$



$$\int_{\mathbb{R}^3} \partial_i j^i = 0$$

LHS: $\displaystyle\int_{\Omega_\varepsilon} d^4x\, \partial_\mu j^\mu(x) = \int d^3x \left(j^0(y^0+\epsilon, \bar{x}) - j^0(y^0-\epsilon, \bar{x}) \right) =$

$$= Q(y^0+\epsilon) - Q(y^0-\epsilon)$$

$$\langle (Q(y^0+\epsilon) - Q(y^0-\epsilon))\, \Phi^i(y) \rangle = \langle 0 |\, T(Q(y^0+\epsilon) - Q(y^0-\epsilon))\, \Phi^i(y)\, | 0 \rangle =$$

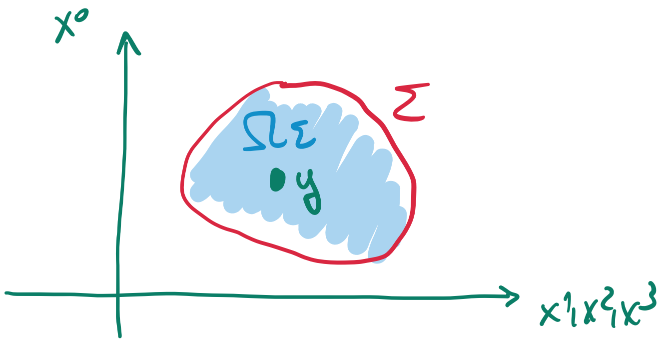
$$= \langle [\hat{Q}(y^0), \hat{\Phi}^i(y)] \rangle \qquad //$$

How does it work for extended objects?

?

Rewriting ordinary sym. transf.

$$i\langle Q(\Sigma)\,\Phi^i(y)\rangle = \mathrm{Link}(\Sigma,y)\, M^i{}_j \langle \Phi^j(y)\rangle$$



Charge Q on a time slice is generalized (Euclidean signature) to a charge $Q(\Sigma)$ on a 3d CLOSED subspace Σ

$$Q(\Sigma) \equiv \int_\Sigma *j$$

The commutation relations to LINK of Σ and y.
How do we derive this relation?

$\downarrow$

Let's integrate w/ $(\circ)$ on Ω_Σ

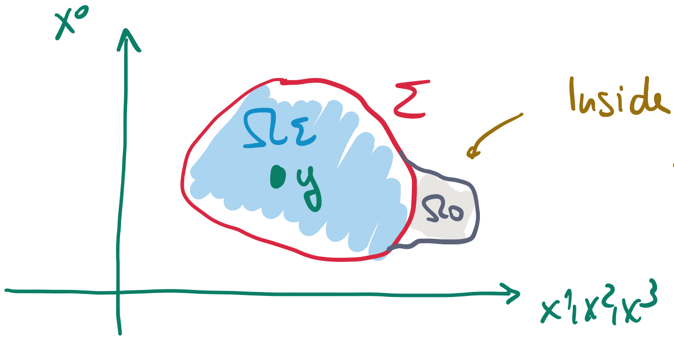
$$\text{LHS}: \int_{\Omega_\Sigma} \partial_\mu j^\mu d^4x = \int_{\Omega_\Sigma} d*j = \int_\Sigma *j = Q(\Sigma)$$

$$\hookrightarrow i\langle Q(\Sigma)\,\Phi^i(y)\rangle = \underbrace{\int_{\Omega_\Sigma} d^4x\, \delta^4(x-y)}_{\mathrm{Link}(\Sigma,y)}\, M^i{}_j \langle \Phi^j(y)\rangle$$

$\leftarrow$ TOPOLOGICAL INVARIANT

Also this is TOPOLOGICAL due to conserv. law:

under a contin. deform. $\Sigma \to \Sigma' = \Sigma + \partial\Omega_0$ $y \in \Omega$



Inside Ω_0 there is NO INSERTION of local operators in correlators

$= 0 \leftrightarrow \partial_\mu j^\mu = 0$

$$Q(\Sigma') = Q(\Sigma) + \int_{\partial\Omega_0} *j = Q(\Sigma) + \int_{\Omega_0} d*j = Q(\Sigma)$$

By exponentiating infinitesimal generators:

$$\langle U_g(\Sigma) \; \Phi^i(y) \rangle = R(g)^i{}_j \; \langle \Phi^j(y) \rangle \qquad \text{(if LINKED)}$$

charged operator (0-dim ⇝ 0-form sym.)

TOPOLOGICAL unitary operator depending on $g \in G$ & Σ

$$\left[\frac{d}{d\alpha} U_{e^{i\alpha}}(\Sigma) \Big|_{\alpha=0} = i\, Q(\Sigma) \right]$$

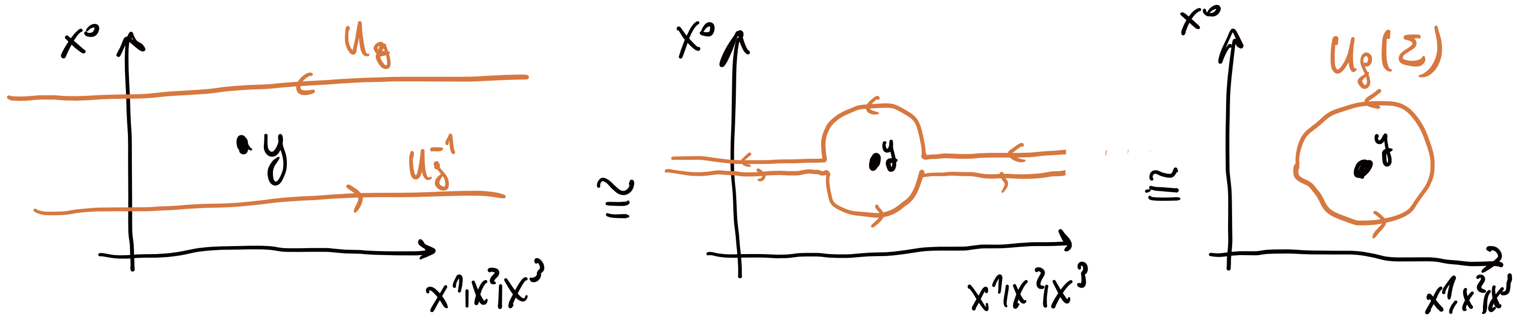
Discrete symmetries

- $g \in G$ discrete
- U_g : unitary operator commuting with Hamiltonian & momentum $[U_g, P^\mu] = 0$
- $\langle U_g \, \Phi^i(y) \, U_g^{-1} \rangle = R(g)^i{}_j \, \langle \Phi^j(y) \rangle$

related to a TOPOL. OPERATOR $U_g(\Sigma)$ s.t.

$$\langle U_g(\Sigma) \, \Phi^i(y) \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle \qquad \text{(if linked)}$$

- $[U_g, P^\mu] = 0 \implies U_g$ can continuously move, i.e. is TOPOL.



- $U_g(\Sigma)\, U_{g'}(\Sigma) = U_{gg'}(\Sigma)$

Summary ORDINARY SYMMETRIES

$g \in G \longleftrightarrow$ TOPOL. OP. $U_g(\Sigma)$ $\qquad$ G cont/disc.

with

$$\langle \underbrace{U_g(\Sigma)}_{\text{topological}}\, \underbrace{\Phi^i(y)}_{\text{not necessary topol.}} \rangle = R^i{}_j(g) \langle \Phi^j(y) \rangle \quad (\text{if linked}) \qquad (*)$$

Σ is $(d-1-\underline{0})$-dim. $\qquad$ $\Phi^i(y)$ is $\underline{0}$-dim $\rightsquigarrow$ "0-form symmetry"

Reduced the problem of finding symmetries to the problem of finding topological operators.

$\downarrow$

This can be generalized to topological $(d-1-p)$-op. i.e. p-form symmetries (we are now going to illustrate an example with $p=1$).

Observation: interpret $(*)$ as

$$\langle U_g(\Sigma)\Phi^i(y)\rangle = R^i{}_j(g)\langle\Phi^j(y)\rangle + 0 = R^i{}_j(g)\langle\Phi^j(y)\rangle + \langle U_g(\Sigma')\Phi^i(y)\rangle$$

with $\mathrm{Link}(\Sigma', y) = 0$

1-form symmetries in Maxwell theory

$$S = -\frac{1}{4e^2}\int F_{\mu\nu}F^{\mu\nu}\, d^4x \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\sim -\int dA \wedge * dA \qquad \rightarrow \text{ e.o.m.} \quad \partial_\mu F^{\mu\nu} = 0 \Leftrightarrow d*F = 0$$

$$\text{Bianchi id.} \quad \partial_\mu * F^{\mu\nu} = 0 \Leftrightarrow dF = 0$$

- There are CONSERVED QUANTITIES
  - Electric flux

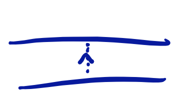
$$Q_E(S^2) = \frac{1}{e^2}\int_{S^2} *F \sim \int_{S^2} \bar{E}\cdot d\bar{S}$$

  - Magnetic flux

$$Q_M(S^2) = \frac{1}{2\pi}\int_{S^2} F \sim \int_{S^2} \bar{B}\cdot d\bar{S}$$

- Both $Q_E(S^2)$ & $Q_M(S^2)$ are TOPOLOGICAL under contin. deformations of S^2.

  $\Longrightarrow$ there should be corresponding symmetries (whose related conserved quantities are the topol. op's)



- Sym. operators:

$$U_E(S^2) \sim e^{iQ_E(S^2)} \qquad U_M(S^2) \sim e^{iQ_M(S^2)}$$

- What are CHARGED OBJECTS? (for ordinary sym it was $\Phi^i(y)$)

  WILSON LOOPS & 't HOOFT LOOPS

- What are GROUPS labelling sym. operators?

U(1)

Wilson Loop $\quad W(q_E,\gamma) = e^{iq_E\int_\gamma A}$ (probe particle)

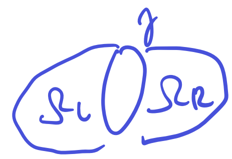
- gauge group $U(1) \ni e^{i\lambda}$ $\quad \lambda \sim \lambda + 2\pi$
- $\lambda(x)$ can have winding number on γ : $\int_\gamma d\lambda = 2\pi w \quad w\in\mathbb{Z}$
- Wilson loop gauge inv. $(A \to A + d\lambda)$

$$e^{iq_E\int_\gamma A} \overset{!}{=} e^{iq_E\int_\gamma A}\, e^{iq_E\int_\gamma d\lambda} \iff q_E 2\pi w \in 2\pi\mathbb{Z} \quad \forall w\in\mathbb{Z}$$
$$\iff q_E \in \mathbb{Z}$$

i.e. Large gauge inv. of WL $\Rightarrow$ DIRAC quantization

[On the other hand,

$$e^{iq_E\int_\gamma A} = e^{iq_E\int_{\Omega_L} F} = e^{iq_E\int_{\Omega_R} F}$$



$$\Rightarrow e^{iq_E\int_{S^2} F} = 1 \qquad S^2 = \Omega_L \cup \bar{\Omega}_R$$

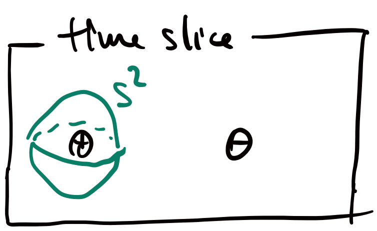
$$\Rightarrow \underbrace{\int_{S^2} F}_{= q_m} \in \frac{2\pi\mathbb{Z}}{q_E} \longrightarrow \text{Dirac quant.} \quad q_E q_m = 2\pi n \quad n\in\mathbb{Z}]$$

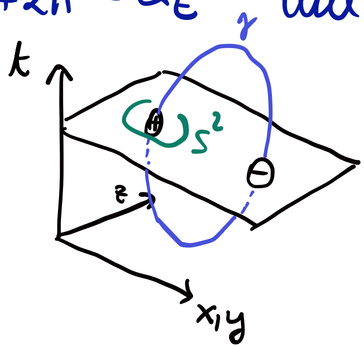
- Symmetry transformation

$$\langle \underbrace{U_{e^{i\alpha_E}}(S^2)}_{= e^{i\alpha_E Q_E(S^2)}}\, e^{iq_E\int_\gamma A} \rangle = e^{i\alpha_E q_E \mathrm{Link}(S^2,\gamma)} \langle e^{iq_E\int_\gamma A} \rangle$$

Link(S^2,γ): Linking number between S^2 and γ

Sym. group is $U(1)$ $\quad \alpha_E + 2\pi \sim \alpha_E$, due to quant. of q_E.





Dim. $\langle U_{e^{i\alpha_E}}(S^2)\, e^{iq_E\int_\gamma A}\rangle = \int \mathcal{D}A\; e^{iS + i\alpha_E Q_E(S^2) + iq_E\int_\gamma A}$

$$Q_E(S^2) = \frac{1}{e^2}\int_{S^2} \star F = \frac{1}{e^2}\int_{B_3} d\star F \equiv 2\int J_{B_3} \wedge d\star F$$

$$S[\overbrace{A - \alpha_E J_{B_3}}^{\equiv A'}] = \int (dA - \alpha_E dJ_{B_3}) \wedge \star(dA - \alpha_E dJ_{B_3}) =$$

$$= S[A] + 2\alpha_E \int J_{B_3} \wedge d\star F + \alpha_E^2 \int dJ_{B_3} \wedge \star dJ_3$$

$= \int_{S^2} \star dJ_3$ It can be regularized by a local counter-term

$$= S[A] + \alpha_E Q_E(S^2) + \dots$$

$$\Rightarrow \langle U_{e^{i\alpha_E}}(S^2)\, e^{iq_E\int_\gamma A}\rangle = \int \mathcal{D}A'\, e^{iS[A']}\, e^{iq_E\int_\gamma A'}\, e^{iq_E\alpha_E\int_\gamma J_{B_3}}$$

$$= e^{iq_E\alpha_3 \underbrace{\int_\gamma J_{B_3}}}\, \langle e^{iq_E\int_\gamma A}\rangle$$

Intersection number between γ and B_3, i.e. Link (S^2, γ)

//

Summary:

- Sym op: $U_{e^{i\alpha_E}}(S^2) = e^{i\alpha_E Q_E(S^2)}$ 2d topol. op.
- Charged op: $e^{iq_E\int_\gamma A}$
- Sym. group: $e^{i\alpha_E} \in U(1)$

$\downarrow$

"ELECTRIC 1-form SYMMETRY"

't HOOFT LOOP $\quad T(q_M, \gamma)$

- Probe magn. part. (monopole)
- Closed line $\Leftrightarrow$ gauge invariance of dual photon
- $q_M \in \mathbb{Z}$ (if $q_E = 1$)
- obtain same formal expression as before when we dualize electric $\longleftrightarrow$ magnetic.

$\downarrow$

"MAGNETIC 1-form SYMMETRY"

Generalisations

G p-form symmetry in d dim :

- Sym op. $U_g(\Sigma_{d-p-1})$
- Charged objects $W(q, \gamma_p)$
- Sym. transf. $\langle U_g(\Sigma_{d-p-1}) W(q,\gamma_p)\rangle = R(g)^q \langle W(q,\gamma_p)\rangle$
  if linked

Take-home message : Existence of sym = Existence of TOPOLOGICAL OPERATORS

1-form symmetries in YM theory

$G = SU(N)$:

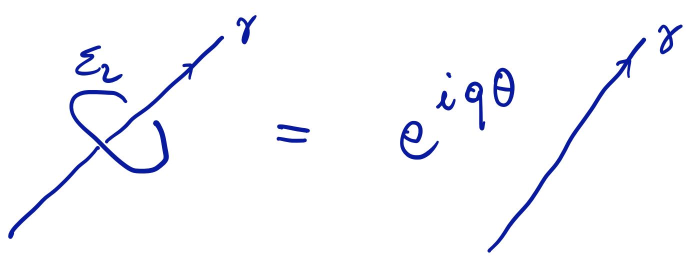
- Wilson lines $W(\gamma)$ can lie in all REPs of G (i.e. charges can be anywhere in weight lattice)

- There exist a SURFACE OPERATOR $U(\Sigma_2)$ that will be the generator of a $\mathbb{Z}_N$ one-form symmetry ; in fact

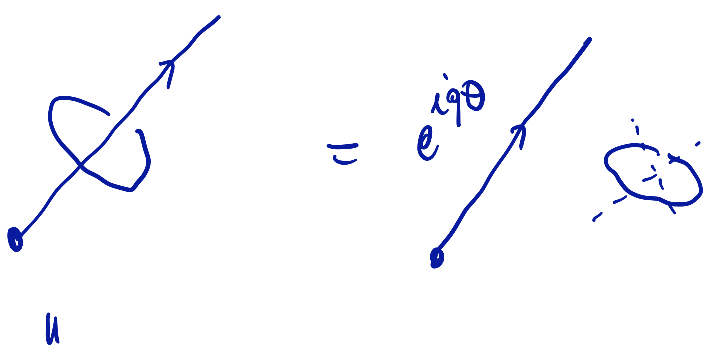
$\langle U(\Sigma_2) W(\gamma) \dots \rangle$ and $\langle W(\gamma) \dots \rangle$ turn out to differ by a factor $e^{2\pi i/N \cdot \mathrm{Link}(\Sigma_2, \gamma)}$

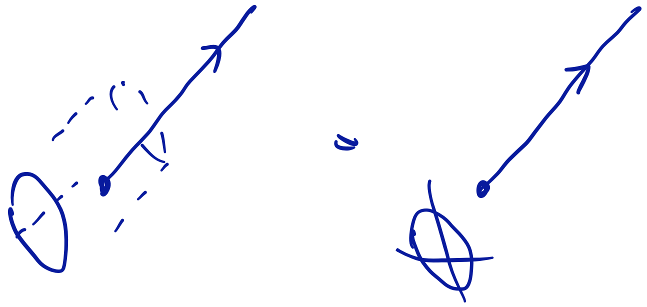
Equivalently $\mathbb{Z}_N$ 1-form sym shifts gauge field by a flat $\mathbb{Z}_N$ gauge connection

Why $\mathbb{Z}_N$? Let's remember how 1-form sym work :



If we now add charged fields, then there will be line that can end on charged particles. Now





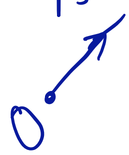
$\implies$ To be consistent:

$$e^{iq\theta} = 1$$

i.e. $\theta = \frac{2\pi k}{q}$

$$U(1)^{(1)} \to \mathbb{Z}_q^{(1)}$$

- In Maxwell theory there is no charged field.
- However, in YM there are ADJOINT PARTICLES, i.e. the gluons.

Only the WL in reps that are not tensors of Adj rep the  is impossible and there can be non trivial transf. associated.

From here we see that we have a $\mathbb{Z}_N^{(1)}$-sym.