Advanced Quantum Mechanics

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Linear Algebra: Vector space LIIICAI AI

DEFINITION 1.1 A vector space V is a set with the following properties;

- (0-1) For any $u, v \in V$, their sum $u + v \in V$.
- (0-2) For any $u \in V$ and $c \in K$, their scalar multiple $cu \in V$.
- $(1-1)$ $(u + v) + w = u + (v + w)$ for any $u, v, w \in V$.
- $(1-2)$ $u + v = v + u$ for any $u, v \in V$.
- (1-3) There exists an element $0 \in V$ such that $u + 0 = u$ for any $u \in V$. This element 0 is called the zero-vector.
- (1-4) For any element $u \in V$, there exists an element $v \in V$ such that $u+v=0$. The vector *v* is called the **inverse** of *u* and denoted by $-u$.

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(2-1) c(x + y) = cx + cy for any c \in K, u, v \in V.
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- $(2-2)$ $(c+d)u = cu + du$ for any $c, d \in K, u \in V$.
- $(2-3)$ $(cd)u = c(du)$ for any $c, d \in K, u \in V$.

(2-4) Let 1 be the unit element of *K*. Then $1u = u$ for any $u \in V$.

Fundamental property: vectors can be stretched and added.

The usual rules of addition and multiplication hold.

There is a null vector.

In $QM: K = \mathbb{C}$ (complex vector space)

Notation
Notation considered.

Vectors will be denoted as follows *x* colors will be actioned as follows

Dirac notation: **ket**
$$
\longleftarrow
$$
 $|x\rangle = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{C}$
Usual notation

It is one of a *n* a *k a x e z x k* a *x e z z x x n x e z z x x n x e z z x x n x e z z x x n x n z x x n x n x n x n x n x* the integral integral integrate. The integration of the dimension of the vector space \mathbf{r} is called the vector space. τ [|]^{*n*} α α β β β β β β β Therefore we have: Therefore we have:

$$
|x\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, |y\rangle = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \Rightarrow |x\rangle + |y\rangle = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}, |a|x\rangle = \begin{pmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_n \end{pmatrix}
$$

(1.2)

Linear (in-)dependence, basis, dimension respectively. All the components of the zero-vector *|*0! are zero. The zerothat these definitions satisfy all the axioms in the axioms in the axioms in the definition of a vector space. *ci|xi*! = *|*0! (1.3)

 $\n **Linear combination**\n $c_1|x\rangle + c_2|y\rangle$$ 2.2 Linear Computation $\sigma_1 | \omega_1 + \omega_2 | \omega_1$

[|]x!*, [|]y*! ∈ ^C*ⁿ* with *^c*1*, c*² [∈] ^C is also an element of ^C*ⁿ*. Linear independent vectors: a set of vectors is linearly independent iff their only linear combination resulting in the null vector can be obtained with all coefficients equal to 0. Otherwise there are called **linearly dependent**. ly linear combination resulting in the null vector can be obtained
coefficients equal to 0. Otherwise there are called **linearly** coefficients equal to U. Otherwise there are called **linearly**

dependent.
\n
$$
\sum_{i=1}^{k} c_i |x_i\rangle = |0\rangle \iff c_i = 0 \ (1 \leq i \leq k)
$$
\nNull vector

Basis: a set of linear independent vectors such that any other vector can be written as linear combination of those vectors. ECCO: infect mappendent tectors sach that any same rector can. be written as linear combination of those vectors.

> **Dimension:** number of basis vectors (n), always for all and α If, in contrast, the trivial solution *cⁱ* = 0 (1 ≤ *i* ≤ *k*) is the only solution of **Dimension:** number of basis vectors (n), always finite for us. Then, V = Cⁿ 3

Examples If, in contrast, the trivial solution *cⁱ* = 0 (1 ≤ *i* ≤ *k*) is the only solution of Eq. (1.3), the set is said to be linearly independent. **EXAMPLES COMPUTER**

EXERCISE 1.1 Find the condition under which two vectors \mathbf{Y} **he cited is an exercise for the readers.** Suppose the readers is a new line-**EAERCISE I.1** FING the condition under which two vectors

$$
|v_1\rangle = \begin{pmatrix} x \\ y \\ 3 \end{pmatrix}, \ |v_2\rangle = \begin{pmatrix} 2 \\ x - y \\ 1 \end{pmatrix} \in \mathbb{R}^3
$$

are linearly independent.

EXERCISE 1.2 Show that a set of vectors

$$
|v_1\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad |v_3\rangle = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}
$$

is a basis of \mathbb{C}^3 .

Inner product

It is a function $\langle .|. \rangle: V \times V \rightarrow \mathbb{C}$

With the following properties:

1.
$$
\langle x | [\alpha | y \rangle + \beta | z \rangle] = \alpha \langle x | y \rangle + \beta \langle x | z \rangle
$$

2. $\langle x | y \rangle = \langle y | x \rangle^*$

3.
$$
\langle x|x\rangle \ge 0
$$
 and is null iff $|x\rangle = |\omega\rangle$

Norm and metric. Hilbert spaces

The inner product defines automatically a **norm**

$$
||x|| = \sqrt{\langle x|x\rangle}
$$

and a **metric** (distance)

$$
d(x,y) = ||x - y||
$$

Hilbert space (ff) **:** a vector space with a inner product (simple definition because we are working with vector spaces of finite dimension)

Linear functionals

It is a function $f: \mathcal{H} \to \mathbb{C}$

such that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

It naturally defines a vector space \mathcal{H}^* , called the **(algebraic) dual** of \mathcal{H} .

 $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

 $(\alpha f)(x) = \alpha f(x)$

Linear functionals

Let $\{\hat{e}_i\}$ with $i = 1,...n$ be a basis of \mathcal{H} . Then for any vector x:

$$
f(x) = f\left(\sum_{i=1}^{n} x_i \hat{e}_i\right) = \sum_{i=1}^{n} x_i f(\hat{e}_i) = \sum_{i=1}^{n} x_i \xi_i \quad \text{with} \quad \xi_i = f(\hat{e}_i) \in \mathbb{C}
$$

Therefore f is uniquely identified by the numbers $(\xi_1,\xi_2, ... \xi_n)$, which are the values of f at the basis vectors. In particular let us consider the functionals

$$
(\xi_1, \xi_2, ..., \xi_n)
$$

\n
$$
(1,0, ... 0) \leftrightarrow \hat{e}_{1}^*
$$

\n
$$
(0,1, ... 0) \leftrightarrow \hat{e}_{2}^*
$$

\n
$$
...
$$

\n
$$
(0,0, ... 1) \leftrightarrow \hat{e}_{n}^*
$$

By construction:
$$
\hat{e}_{i}^{*}(\hat{e}_{j}) = \delta_{ij}
$$

It can be shown that $\{\hat{e}_{i}\}$ forms a basis of \mathcal{H}^* called the **dual basis**

Riesz's representation theorem

Every functional on $\mathcal H$ can be represented in terms of an inner product $f(x) = \langle z|x \rangle$

where z depends on f, and is uniquely determined by it. Therefore

$$
f \leftrightarrow z
$$
 such that $f(\cdot) = \langle z | \cdot \rangle$

There is a **1-to-1 correspondence between vectors and functionals.**

Dirac notation case) is called the dual vector space, or simply the dual space, of *V*

Given a basis |1>, |2> … |n> in , we *Basics of Vectors and Matrices* 5 will always consider the dual basis of $\boldsymbol{\mathcal{H}}^*$, \vert \vert \vert which we will denote as <1|, <2| ... <n|. Then and the called a ket vector or simply a ket vector or simply a ket. We will later the simply a ket vector or simply a ket vector o

$$
\langle i | j \rangle = \delta_{ij}.
$$

Also Also Also important linear function is a bra vector obtained from a bra vector obtained from a ket vector. The set o

$$
|x\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ \mapsto \langle x| = (x_1^*, \dots, x_n^*) \quad \text{so that} \qquad \langle x|(|y\rangle) = \langle x|y\rangle = \sum_{i=1}^n \overbrace{x_i^*}^{Riesz's} y_i
$$

This gives a clear mathematical meaning to the Dirac bra-ket notation This gives a **clear mathematical meaning to the Dirac bra-ket notation**

Example

EXERCISE 1.3 Let

$$
|x\rangle = \begin{pmatrix} 1 \\ i \\ 2+i \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} 2-i \\ 1 \\ 2+i \end{pmatrix}
$$

Find $|||x\rangle||$, $\langle x|y\rangle$ and $\langle y|x\rangle$.

α are called the components of α β bination of these basis vectors as *[|]x*! ⁼ %*ⁿ* A basis *{|ei*!*}* that satisfies

 $\langle e_i | e_j \rangle = \delta_{ij}$ $i \circ \alpha = \alpha$

Let $|x\rangle = \sum_{i=1}^{n} c_i |e_i\rangle$. The inner product of $|x\rangle$ and $\sum_{i=1}^{\infty} \frac{1}{i}$ $\sum_{i=1}^{\infty} \frac{1}{i}$ Let $|x\rangle = \sum_{i=1}^{n} c_i |e_i\rangle$. The inner product of $|x\rangle$ and $\langle e_j |$ yields

$$
\langle e_j|x\rangle = \sum_{i=1}^n c_i \langle e_j|e_i\rangle = \sum_{i=1}^n c_i \delta_{ji} = c_j \rightarrow c_j = \langle e_j|x\rangle.
$$

Linear Operator Linear Operator 1.5 Linear Operators and Matrices is satified for arbitrary *[|]x*!*, [|]y*! ∈ ^C*ⁿ* and *^c^k* [∈] ^C. Let us choose an arbitrary A map *^A* : ^C*ⁿ* [→] ^C*ⁿ* is a linear operator if orthonormal basis *{|ek*!*}*. It is shown below that *A* is expressed as an *n* × *n* r ator provided that its action on the basis vectors is given. Since *^A|ek*! ∈ ^C*ⁿ*, it

A map $A: \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator if n_{ice} linear eperat

 $A(c_1|x\rangle +c_2|y\rangle)=c_1A|x\rangle +c_2A|y\rangle$

is satified for arbitrary $|x\rangle, |y\rangle \in \mathbb{C}^n$ and $c_k \in \mathbb{C}$. Let us choose an arbitrary orthonormal basis $\{ |e_k \rangle \}$. It is shown below that *A* is expressed as an $n \times n$ matrix. $\frac{1}{e^{k}}$. To be another vector in C_n *^k vkA|ek*!. Therefore, the action of *A* on an arbitrary vector is fixed can be expanded to the expansion of *n* $\{e_k\}$. It is shown below that A is expressed as an $n \times n$

$$
A|e_k\rangle = \sum_{i=1}^n |e_i\rangle A_{ik}. \qquad A_{jk} = \langle e_j|A|e_k\rangle. \qquad A = \sum_{j,k} A_{jk}|e_j\rangle\langle e_k|
$$

Projection Operator Dr_o The *matrix*

 $P_k \equiv |e_k\rangle\langle e_k|$ $\mathbf{1} \kappa - |\mathcal{C}\mathcal{K}| \setminus \mathcal{C}\mathcal{K}|$

The set $\{P_k = |e_k\rangle\langle e_k|\}$ satisfies the conditions $\mathbf{1} \mathcal{R} = |\mathcal{C}\mathcal{R}| \setminus \mathcal{C}\mathcal{R}|$ The set $\{P_k = |e_k\rangle\langle e_k| \}$ satisfies the conditions $\int P(x) dx \leq |e_x|/e_x$ is orticfied $\text{The set } { \{P_k \} } \subseteq { |e_k \rangle \langle e_k | }$ satisfies

(i)
$$
P_k^2 = P_k
$$
,
\n(ii) $P_k P_j = 0 \quad (k \neq j)$,
\n(iii) $\sum_k P_k = I \quad \text{(completeness relation)}$.

EXAMPLE 1.1 Let

(i) *P*²

EXAMPLE 1.1 Let

$$
|e_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |e_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
$$

$$
\sum_k P_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I
$$

 Γ *P*² (*I.e. iii)*
introduced above is easily verified. Projection operators and the orthogonality condition $1/1 \ 1)$ 1 They define an orthonormal basis as is easily verified. Projection operators and the orthogonality condition are

are
\n
$$
P_1 = |e_1\rangle\langle e_1| = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, P_2 = |e_2\rangle\langle e_2| = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
$$
\n
$$
P_1 P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$
\nThey satisfy the completeness relation
\nThe reader should verify that $P_1^2 = P_k$.

bey satisfy the completeness relation The reader should verify that $P_k^2 = P_k$. α = β

Hermitian Conjugate – Hermitian operator the relation *Ajk* = (*A†*)[∗] *kj* , namely *kj .* (1.24) (*A†*)*jk* = *A*[∗] *kj .* (1.24)

DEFINITION 1.2 (Hermitian conjugate) Given a linear operator *A* : $\mathbb{C}^n \to \mathbb{C}^n$, its Hermitian conjugate A^{\dagger} is defined by $\mathbb{C}^n \to \mathbb{C}^n$, its Hermitian conjugate A^{\dagger} is defined by the complex conjugation of *A*.

$$
\langle u|A|v\rangle \equiv \langle A^\dagger u|v\rangle = \langle v|A^\dagger|u\rangle^*,
$$

where $|u\rangle, |v\rangle$ are arbitrary vectors in \mathbb{C}^n .

(*cA*)

The above definition shows that $\langle e_j | A | e_k \rangle = \langle e_k | A^{\dagger} | e_j \rangle^*$. Therefore, we find the relation $A_{jk} = (A^{\dagger})_{kj}^*$, namely

$$
(A^{\dagger})_{jk} = A^*_{kj}.
$$

$$
(cA)^{\dagger} = c^*A^{\dagger}, \quad (A+B)^{\dagger} = A^{\dagger} + B^{\dagger}, \quad (AB)^{\dagger} = B^{\dagger}A^{\dagger}.
$$

 T **This definition** and $\boldsymbol{\theta}$ (it we have applied $\boldsymbol{\theta}$ and $\boldsymbol{\theta}$ and $\boldsymbol{\theta}$ be a **Hermitian matrix** if it satisifies $A^{\dagger} = A$. **DEFINITION 1.3** (Hermitian matrix) A matrix $A: \mathbb{C}^n \to \mathbb{C}^n$ is said to

Unitary operator 13 and 13 a as a Hermitian conjugation of the ket vector.

DEFINITION 1.4 (Unitary matrix) Let $U: \mathbb{C}^n \to \mathbb{C}^n$ be a matrix which satisfies $U^{\dagger} = U^{-1}$. Then *U* is called a **unitary matrix**. Moreover, if *U* is unimodular, namely det $U = 1$, U is said to be a **special unitary matrix**.

The set of unitary matrices is a group called the **unitary group**, while that of the special unitary matrices is a group called the special unitary **group**. They are denoted by $U(n)$ and $SU(n)$, respectively.

Let $\{|e_1\rangle, \ldots, |e_n\rangle\}$ be an orthonormal basis in \mathbb{C}^n . Suppose a matrix U : $\mathbb{C}^n \to \mathbb{C}^n$ satisifes $U^{\dagger}U = I$. By operating U on $\{|e_k\rangle\}$, we obtain a vector $|f_k\rangle = U|e_k\rangle$. These vectors are again orthonormal since

$$
\langle f_j | f_k \rangle = \langle e_j | U^{\dagger} U | e_k \rangle = \langle e_j | e_k \rangle = \delta_{jk}.
$$
 (1.26)

Note that $|\det U| = 1$ since $\det U^{\dagger}U = \det U^{\dagger} \det U = |\det U|^2 = 1$.

Eigenvalues & Eigenvectors Eigenvalues & Eigenvectors \blacksquare corresponding to an eigenvalue \blacksquare Let *{|ek*"*}* be an orthonormal basis in ^C*ⁿ* and let &*ei|A|e^j* " ⁼ *^Aij* and *vⁱ* = &*ei|v*" be the components of *A* and *|v*" with respect to the basis. Then

eigen equation of *A*.

 $A|v\rangle = \lambda|v\rangle, \quad \lambda \in \mathbb{C}.$ $t_{\text{max}} = \frac{1}{2}$

Then λ is called an **eigenvalue** of *A*, while $|v\rangle$ is called the corresponding eigenvector. The above equation being a linear equation, the norm of the eigenvector cannot be fixed. Of course, it is always possible to normalize $|v\rangle$ such that $|||v\rangle|| = 1$. We often use the symbol $|\lambda\rangle$ for an eigenvector corresponding to an eigenvalue λ to save symbols corresponding to an eigenvalue λ to save symbols. *A*^{*k*} an eigenvalue of *A*, while $|v\rangle$ is called the

Let *{|ek*"*}* be an orthonormal basis in ^C*ⁿ* and let &*ei|A|e^j* " ⁼ *^Aij* and Let us find the eigenvalue λ next. Note first that the eigenvalue equation is rewritten as rewritten as 14 *QUANTUM COMPUTING* $\sum (A - \lambda I) \cdot u = 0$

$$
\sum_{j} (A - \lambda I)_{ij} v_j = 0.
$$

i,j i,j has no inverse, namely This equation in v_j has nontrivial solutions if and only if the matrix $A - \lambda I$ If it had the inverse, then *[|]v*# = (*^A* [−] ^λ*I*)−¹*|*0# = 0 would be the unique

$$
D(\lambda) \equiv \det(A - \lambda I) = 0.
$$
 characteristic equation

Eigenvalues & Eigenvectors of Hermitian operators of Hermitian matrices and unitary matrices and unitary matrices are particles and unitary matrices are paraoperators. The practical applications. The eigenvalue problems of Hermitian matrices and unitary matrices and unitary matrices are part- T eigenvalue problems of Hermitian matrices and unitary matrices and unitary matrices are partticularly in proportions. The eigenvalue problems of Hermitian matrices and unitary matrices are par-*P*+*|* ↑# = *|* ↑#*, P*+*|* ↓# = 0*, P*−*|* ↑# = 0*, P*−*|* ↓# = *|* ↓# *.* $f \alpha r c$

THEOREM 1.2 All the eigenvalues of a Hermitian matrix are real numbers. Moreover, two eigenvectors corresponding to different eigenvalues are orthogonal. THEOREM 1.2 All the eigenvalues of a Hermitian matrix are real numbers. Moreover, two eigenvectors corresponding to different eigenvalues are orthogonal. The eigenvalues of a Hermitian matrix are real number of a Hermitian matrix are real number of a Hermitian matrix are real num-**THEOREM 1.2** All the eigenvalues of a Hermitia Ortnogonal. *Property and a be a letter matrix and letter and letter* $\mathbf{F} = \mathbf{F} \cdot \mathbf{F}$ = $\mathbf{F} = \mathbf{F} \cdot \mathbf{F}$

Proof. Let *A* be a Hermitian matrix and let $A|\lambda\rangle = \lambda|\lambda\rangle$. The Hermitian conjugate of this equation is $|\lambda|A = \lambda^*/\lambda|$. From these equations we obtain conjugate of this equation is $\langle \lambda | A = \lambda^* \langle \lambda |$. From these equations we obtain $\langle \lambda | A | \lambda \rangle = \lambda \langle \lambda | \lambda \rangle = \lambda \langle \lambda | \lambda \rangle$, which proves $\lambda = \lambda$ since $\langle \lambda | \lambda \rangle \neq 0$.
Let $A | \mu \rangle = \mu | \mu \rangle$ $(\mu \neq \lambda)$ next Then $\langle \mu | A = \mu | \mu \rangle$ since μ . Let $A|\mu\rangle = \mu|\mu\rangle$ ($\mu \neq \lambda$), next. Then $\langle \mu | A = \mu \langle \mu |$ since $\mu \in \mathbb{R}$. From
 $|A|\lambda\rangle = \lambda/\mu |\lambda\rangle$ and $|\mu|A|\lambda\rangle = \mu/\mu |\lambda\rangle$ we obtain $0 = (\lambda - \mu)/\mu |\lambda\rangle$. Since $\langle \mu | A | \lambda \rangle = \lambda \langle \mu | \lambda \rangle$ and $\langle \mu | A | \lambda \rangle = \mu \langle \mu | \lambda \rangle$, we obtain $0 = (\lambda - \mu) \langle \mu | \lambda \rangle$. Since $\mu \neq \lambda$ we must have $\langle \mu | \lambda \rangle = 0$. $\mu \neq \lambda$, we must have $\langle \mu | \lambda \rangle = 0$. *Proof.* Let A be a Hermitian matrix and let $A|\lambda\rangle = \lambda|\lambda\rangle$. The Hermitian conjugate of this equation is $\langle \lambda | A = \lambda^* \langle \lambda |$. From these equations we obtain $\langle \lambda | A | \lambda \rangle = \lambda \langle \lambda | \lambda \rangle = \lambda^* \langle \lambda | \lambda \rangle$, which proves $\lambda = \lambda^*$ since $\langle \lambda | \lambda \rangle \neq 0$. Let $A|\mu\rangle = \mu|\mu\rangle$ $(\mu \neq \lambda)$, next. Then $\langle \mu|A = \mu \langle \mu|$ since $\mu \in \mathbb{R}$. From $\langle \mu | A | \lambda \rangle = \lambda \langle \mu | \lambda \rangle$ and $\langle \mu | A | \lambda \rangle = \mu \langle \mu | \lambda \rangle$, we obtain $0 = (\lambda - \mu) \langle \mu | \lambda \rangle$. Since $\mu \neq \lambda$, we must have $\langle \mu | \lambda \rangle = 0$. $\frac{1}{4}$ $\frac{1}{2}$ $\frac{1}{2}$ $\text{conjugate of this equation is } \langle A | A = \lambda^\top \langle A | A \rangle$ tion, for example, to obtain an orthonormal basis in this *k*-dimensional space. $\mathcal{A}|\Lambda\rangle = \Lambda|\Lambda\rangle$. The Hermitian $\mathbf{P}^{\mathbf{P}}$ $\mathbf{P}^{\mathbf{P}}$ and $\mathbf{P}^{\mathbf{P}}$ and $\mathbf{P}^{\mathbf{P}}$ are obtain $\mathbf{P}^{\mathbf{P}}$ tion, for example, to obtain an orthonormal basis in this *k*-dimensional space. \mathbf{A}_c and set of eigenvectors of a Hermitian matrix is always chosen $\langle \mu | A | \lambda \rangle = \lambda \langle \mu | \lambda \rangle$ and $\langle \mu | A | \lambda \rangle = \mu \langle \mu | \lambda \rangle$, we obta which is called the spectral decomposition of \mathcal{M} .

Experience the set of eigenvectors $\{|\lambda_k\rangle\}$ of a Hermitian matrix A may be made into a complete set Suppose λ is *k*-fold degenerate. Then there are *k* independent eigenvectors Therefore, the set of eigenvectors $\{|\lambda_k\rangle\}$ of a Hermitian matrix *A* may be made into a complete set to be orthonormal. Therefore, the set of eigenvectors *{|*λ*k*#*}* of a Hermitian $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$ and eigenvalues $\frac{1}{\sqrt{2}}$ Therefore, the set of eigenvectors $\{\ket{\lambda_k}\}$ of a Hermitian matrix A may be made into a complete set

$$
A = \sum_{i} \lambda_i |\lambda_i\rangle\langle\lambda_i|, \quad \text{spectral decomposition of } A
$$

k=1 *|*λ*k*#%λ*k|* = *I* $\mathbf{y} = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|, \ \ \vert \ \textbf{ spectral decomposition of } A.$

Exercises

EXERCISE 1.9 Let which proves that &λ*k|*λ*^j* % = 0 for λ*^j* '= λ*k*.

$$
A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}.
$$

Find the eigenvalues and the corresponding normalized eigenvectors. Show that the eigenvectors are mutually orthogonal and that they satisfy the completeness relation. Find a unitary matrix which diagonalizes *A*. Find the circuit line and Ing the eigenvalues and the corresponding normalized eigenvectors, show matrice eigenvectors are matematy or inogener and their they pleteness relation. Find a unitary matrix which diagonalizes A. *Basic dia Grand die Corresponding normanzed eigenvectors.* Show

EXERCISE 1.10 (1) Suppose *A* is skew-Hermitian, namely $A^{\dagger} = -A$. Show that all the eigenvalues are pure imaginary. Show that all the eigenvalues are pure. RCISE 1.10 (1) Suppose A is skew-He

- (2) Let U be a unitary matrix. Show that all the eigenvalues are unimodular, namely $|\lambda_i| = 1$. ultix. dhov $\overline{}$ \overline{a} \overline{b} \overline{a} \overline{b} \overline{a} \overline{b} $\overline{\Omega}$ has real eigenvalues *±* [√]*a*² [−] *^b*² for *[|]a[|]* [≥] *[|]b|*. How about the orthonormality
- (3) Let *A* be a normal matrix. Show that *A* is Hermitian if and only if all the eigenvalues of *A* are real. *a* for *a* are real.
This of *A* are real. of the eigenvectors? Let A be a normal

A matrix *A* is normal if it satisfies $AA^{\dagger} = A^{\dagger}A$

Exercise 2.12: Prove that the matrix **b**rancise 2.12. Draw that the matrix Γ ¹ and Γ ² and Γ Γ ¹ and Γ ¹ and it is easily checked that it is easily checked that Γ ¹ and Γ

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

is not diagonalizable.

 2.1 and ω and ω and ω **Exercise 2.13:** If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^{\dagger} = |v\rangle\langle w|$

can be written in the form *eⁱ*^θ for some real θ.

leges) Suppose A' and A'' are matrix representations operator *A* on a vector space *V* with respect to two different orthonormal bases, $|v_i\rangle$ and $|w_i\rangle$. I hen the elements of *A* and *A* are $A_{ij} - \langle v_i|A|v_j\rangle$ and $A'' = \langle v_i | A | w_i \rangle$. Characterize the relationship between A' and A'' . $A_{ij}^{\prime\prime} \equiv \langle w_i | A | w_j \rangle$. Characterize the relationship between A' and A''. $\overline{}$ $A_{ij}^{\prime\prime} = \langle w_i | A | w_j \rangle$. Characterize the relationship between *A*' and *A*". *a*^{*i*} *a*_{*i*} *<i>a***^{***i***} ***a*_{*i*} *<i>a <i>a***^{***i***} ***<i>a <i>a*^{*i*} *<i>a <i>a*^{*i*} *a a*^{*i*} Exercise 2.15: Show that (*A†*) Exercise 2.20: (Basis changes) Suppose A' and A'' are matrix representations of an $|v_i\rangle$ and $|w_i\rangle$. Then the elements of *A*' and *A*" are $A'_{ij} = \langle v_i | A | v_j \rangle$ and

Pauli matrices et al. et a The Pauli matrices, also known as the spin matrices, also known as the spin matrices, are defined by an are defined by an are defined by a spin matrices, and the spin matrices, and the spin matrices, and the spin matrices, even though the terminology of spin algebra may be employed. often employ natural units in which is the tracelessness property in which is not traceless property in the tra
In the tracelessness property in the tracelessness property in the tracelessness property in the tracelessness

$$
\sigma_x = \left(\begin{matrix} 0 \ 1 \\ 1 \ 0 \end{matrix}\right), \quad \sigma_y = \left(\begin{matrix} 0 \ -i \\ i \ 0 \end{matrix}\right), \quad \sigma_z = \left(\begin{matrix} 1 \ 0 \\ 0 \ -1 \end{matrix}\right).
$$

They are also referred to as σ1*,* σ² and σ3, respectively. **THE SYMBOL CONDUCT AND INCONDUCT OF ORDER I**
The unit matrix of order $\boldsymbol{\theta}$ and $\boldsymbol{\theta}$ and discussed on the discussed of order $\boldsymbol{\theta}$ and $\boldsymbol{\theta}$ and $\boldsymbol{\theta}$ and $\boldsymbol{\theta}$ are discussed on the discussed of $\boldsymbol{\theta}$ we introduce the unit matrix \mathbf{I} in the algebra, which amounts to expanding the algebra, which amounts to expanding the algebra, which amounts to expanding the second was algebra, which amounts to expanding the second rique de Lie algebra superiorment matrices superiorment matrices satisfy the anticommutation of anticommutation Product of Pauli matrices

$$
\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij} I.
$$

\n
$$
[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2i \sum_k \varepsilon_{ijk} \sigma_k, \qquad \varepsilon_{ijk} = \begin{cases} 1, (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2) \\ -1 (i, j, k) = (2, 1, 3), (1, 3, 2), (3, 2, 1) \\ 0 \text{ otherwise.} \end{cases}
$$

\n
$$
\sigma_i \sigma_j = i \sum_{k=1}^3 \varepsilon_{ijk} \sigma_k + \delta_{ij} I.
$$

Pauli matrices ε*ijk* = \mathbf{r} L 1*,* (*i, j, k*) = (1*,* 2*,* 3)*,*(2*,* 3*,* 1)*,*(3*,* 1*,* 2) The spin-flip ("ladder") operators are defined by −1 (*i, j, k*) = (2*,* 1*,* 3)*,*(1*,* 3*,* 2)*,*(3*,* 2*,* 1)

The spin-flip ("ladder") operators are defined by be more precise, these are expressions that are relevant when the *z*-component The spin mp $($ reduct $)$ operators are defined by The spin-flip ("ladder") operators are defined h ¹

$$
\sigma_+ = \frac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
$$

$$
|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{Eigenstates of } \sigma_z
$$

 $\text{Verify that } \sigma \downarrow \uparrow \rangle = \sigma \downarrow \downarrow \rangle = 0 \quad \sigma \downarrow \downarrow \rangle = \downarrow \uparrow \rangle \quad \sigma \downarrow \uparrow \rangle$ projection operators to the eigenspaces of σ_z with the eigenvalues ± 1 are
 $P_{\alpha} = |\uparrow \rangle / \uparrow | = \frac{1}{2}(I + \sigma_z) = \binom{1}{1}$ Verify that $\sigma_+|\uparrow\rangle = \sigma_-|\downarrow\rangle = 0$, $\sigma_+|\downarrow\rangle = |\uparrow\rangle$, $\sigma_-|\uparrow\rangle = |\downarrow\rangle$. The *PP*^{*P*} \langle *PPPP*^{*P*} \langle *P*_{*P*}^{*P*} \langle *P*

$$
P_{+} = |\uparrow\rangle\langle\uparrow| = \frac{1}{2}(I + \sigma_{z}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
$$

$$
P_{-} = |\downarrow\rangle\langle\downarrow| = \frac{1}{2}(I - \sigma_{z}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

 $\sigma_{\pm} = 0$, $I_{\pm} = I_{\pm}$, $I_{+} = 0$. \overline{a} natural units in \overline{a} . Note the tracelessness property $\sigma_{\pm}^2 = 0$, $P_{\pm}^2 = P_{\pm}$, $P_+P_- = 0$. $\sigma_{\pm}^{2} = 0$, $P_{\pm}^{2} = P_{\pm}$, $P_{+}P_{-} = 0$.

Function of an operator matrix is evaluated quite easily. Let us prove the following formula. 18 *QUANTUM COMPUTING*

PROPOSITION 1.1 Let *A* be Hermitian matrix in the above theorem. Then for an arbitrary $n \in \mathbb{N}$, we obtain −1

$$
A^n = \sum_{\alpha} \lambda_{\alpha}^n P_{\alpha}.
$$

If, furthermore, A^{-1} exists, the above formula may be extended to $n \in \mathbb{Z}$ by noting that λ_{α}^{-1} is an eigenvalue of A^{-1} .

Proof. Let $n \in \mathbb{N}$. Then

$$
A^n P_\alpha = \lambda_\alpha A^{n-1} P_\alpha = \ldots = \lambda_\alpha^{n-1} A P_\alpha = \lambda_\alpha^n P_\alpha,
$$

from which we obtain

$$
A^n = A^n \sum_{\alpha} P_{\alpha} = \sum_{\alpha} A^n P_{\alpha} = \sum_{\alpha} \lambda_{\alpha}^n P_{\alpha}.
$$

To prove the second half of the proposition, we only need to show that A^{-1} has an eigenvalue λ_{α}^{-1} , provided that A^{-1} exists (and hence $\lambda_{\alpha} \neq 0$), and the corresponding projection operator is $P_α$. We find $\frac{1}{2}$ of the proposition, we
vided that A^{-1} evists biny need to snow that

$$
|\lambda_{\alpha,p}\rangle = A^{-1}A|\lambda_{\alpha,p}\rangle = \lambda_{\alpha}A^{-1}|\lambda_{\alpha,p}\rangle \to A^{-1}|\lambda_{\alpha,p}\rangle = \lambda_{\alpha}^{-1}|\lambda_{\alpha,p}\rangle.
$$

Therefore the projection operator corresponding to the eivengalue λ_{α}^{-1} is P_{α} . The case $n = 0$, $I = \sum_{\alpha} P_{\alpha}$, is nothing but the completeness relation. Now we have proved that Eq. (1.42) applies to an arbitrary $n \in \mathbb{Z}$. They may represent any two mutually orthogonal states, such as horizontally orthogonal states, such as horizontal efore the projection operator corresponding to the eivengalue λ_{α}^{-1} : niently expressed in terms of the Pauli matrices σ*^k* as *S^k* = (!*/*2)σ*k*. We

Exercises decomposition is not unique in this case. Of course this ambiguity originates the course this ambiguity or originates in the choice of the branch in the definition of √*x*. L ACT CISCS again. It follows directly from Example L

EXAMPLE 1.6 Let us consider σ_y again. It follows directly from Example 1.5 that EXAMPLE 1.6 Let us consider σ again. It follows directly from Example 1.5 that $\mathbf{F} \mathbf{Y} \Lambda \mathbf{N}$

$$
\exp(i\alpha \sigma_y) \equiv \sum_{k=0}^{\infty} \frac{(i\alpha \sigma_y)^k}{k!} = e^{i\alpha} P_1 + e^{-i\alpha} P_2 = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.
$$

EXERCISE 1.13 Suppose a 2 × 2 matrix *A* has eigenvalues −1*,* 3 and the corresponding eigenvectors EXERCISE 1.13 Suppose a 2 × 2 matrix *A* has eigenvalues −1*,* 3 and the corresponding eigenvectors EXERCISE 1.13 Suppose a 2 × 2 matrix *A* has eigenvalues −1*,* 3 and the $EXERCISE$ 1.13 Suppos

$$
|e_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ i \end{pmatrix}, |e_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix},
$$

respectively. Find *A*. respectively. Find *A*. respectively. Find *A*.

 E EXERCISE 1.14 Let

$$
A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
$$

(1) Find the eigenvalues and the corresponding normalized eigenvectors of *A*.

.

(2) Write down the spectral decomposition of *A*. (1) Find the eigenvalues and the corresponding normalized eigenvectors of *A*. (2) Write down the spectral decomposition of *A*. (2) Write down the spectral decomposition of *A*.

(3) Find $\exp(i\alpha A)$. (3) Find exp(*i*α*A*). (3) Find $\exp(i\alpha A)$.

Exercises L A C I C I S C C

EXERCISE 1.15 Let

$$
A = \begin{pmatrix} 5 & -2 & -4 \\ -2 & 2 & 2 \\ -4 & 2 & 5 \end{pmatrix}.
$$

(1) Find the eigenvalues and the corresponding eigenvectors of *A*.

 (2) Find the spectral decomposition of \overrightarrow{A} .

(3) Find the inverse of *A* by making use of the spectral decomposition.

PROPOSITION 1.2 Let $\hat{\boldsymbol{n}} \in \mathbb{R}^3$ be a unit vector and $\alpha \in \mathbb{R}$. Then

$$
\exp(i\alpha \hat{\boldsymbol{n}}\cdot \boldsymbol{\sigma}) = \cos \alpha I + i(\hat{\boldsymbol{n}}\cdot \boldsymbol{\sigma})\sin \alpha,
$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$.

Exercises since *Z* has eigenvectors *|*0# and *|*1#.

Exercise 2.34: Find the square root and logarithm of the matrix

$$
\left[\begin{array}{cc} 4 & 3 \\ 3 & 4 \end{array}\right].
$$

Tensor product 1.10 Tensor Product (Kronecker Product)

DEFINITION 1.5 Let *A* be an $m \times n$ matrix and let *B* be a $p \times q$ matrix. Then **DEFINITION 1.5** Let A be an $m \times n$ matrix and let B Then $(a_{11}B a_{12}B a_{13}B)$

$$
A \otimes B = \begin{pmatrix} a_{11}B, a_{12}B, \dots, a_{1n}B \\ a_{21}B, a_{22}B, \dots, a_{2n}B \\ \dots \\ a_{m1}B, a_{m2}B, \dots, a_{mn}B \end{pmatrix}
$$
 (1.47)

 $\lim_{h \to 0} \frac{1}{h} \times (nq)$ matrix called the **tensor product** (**Kronecker product**) of *A* and *B*. of A and B .

It should be noted that not all $(mp) \times (nq)$ matrices are tensor products of an $m \times n$ matrix and a $p \times q$ matrix. In fact, an $(mp) \times (np)$ matrix has *mnpq* degrees of freedom, while $m \times n$ and $p \times q$ matrices have $mn + pq$ in total. Observe that $mnpq \gg mn + pq$ for large enough m, n, p and q . This fact is ultimately related to the power of quantum computing compared to its $\frac{1}{2}$ classical counterpart.

Exercises and the power of EXEICISES .

EXAMPLE 1.8

$$
\sigma_x \otimes \sigma_z = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
$$

EXAMPLE 1.9 We can also apply the tensor product to vectors as a special case. Let

$$
|u\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \quad |v\rangle = \begin{pmatrix} c \\ d \end{pmatrix}.
$$

Then we obtain

$$
|u\rangle \otimes |v\rangle = \begin{pmatrix} a|v\rangle \\ b|v\rangle \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.
$$

The tensor product $|u\rangle \otimes |v\rangle$ is often abbreviated as $|u\rangle|v\rangle$ or $|uv\rangle$ when it does not cause confusion.

$$
\begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \times 2 \\ 1 \times 3 \\ 2 \times 2 \\ 2 \times 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix}
$$

$$
X \otimes Y = \begin{bmatrix} 0 \cdot Y & 1 \cdot Y \\ 1 \cdot Y & 0 \cdot Y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}
$$

Figure 1.20. Externally, $\frac{1}{2}$, $\frac{1}{2}$, and using the Kroneck $t = 2.25 \pm 0.11$ and $t = 0.1$ and $t = 0.1$ **Exercise 2.26:** Let $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explic in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product. $\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both

1

 \mathbf{I} \mathbb{R}

. (2.52)

on tensor product spaces. Exercise 2.26: Let *|*ψ% = (*|*0% + *|*1%)*/* √ Exercise 2.26: Let *|*ψ% = (*|*0% + *|*1%)*/* commutative? 2. Write out *∂*
ψ%⊗3 explicitly, both out *|*ψ%⊗3 explicitly, both out *|*ψ%⊗3 explicitly, both out *|* on tensor product spaces. √ \mathbf{R} \mathbf{Z} , (0) *I* and \mathbf{A} , (c) \mathbf{A} and *I*. Is the tensor product in terms of tensor products like *|*0%*|*1%, and using the Kronecker product. Exercise 2.27: Calculate the matrix representation of the tensor products of the Pauli operators (a) *X* and *Z*; (b) *I* and *X*; (c) *X* and *I*. Is the tensor product

Exercises *Basics of Vectors and Matrices* ²⁷

EXERCISE 1.18 Let *A* and *B* be as above and let *C* be an $n \times r$ matrix and *D* be a $q \times s$ matrix. Show that

$$
(A \otimes B)(C \otimes D) = (AC) \otimes (BD).
$$

It similarly holds that

$$
(A_1 \otimes B_1)(A_2 \otimes B_2)(A_3 \otimes B_3) = (A_1 A_2 A_3) \otimes (B_1 B_2 B_3),
$$

and its generalizations whenever the dimensions of the matrices match so that the products make sense.

EXERCISE 1.19 Show that

$$
A \otimes (B + C) = A \otimes B + A \otimes C
$$

$$
(A \otimes B)^{\dagger} = A^{\dagger} \otimes B^{\dagger}
$$

$$
(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}
$$

whenever the matrix operations are well-defined.

Show, from the above observations, that the tensor product of two unitary matrices is also unitary and that the tensor product of two Hermitian matrices is also Hermitian.

EXERCISE 1.20 Let *A* and *B* be an $m \times m$ matrix and a $p \times p$ matrix, respectively. Show that

$$
tr(A \otimes B) = (trA)(trB),
$$

$$
det(A \otimes B) = (det A)^p (det B)^m.
$$

Exercises

EXERCISE 1.21 Let $|a\rangle, |b\rangle, |c\rangle, |d\rangle \in \mathbb{C}^n$. Show that

 $(|a\rangle\langle b|) \otimes (|c\rangle\langle d|) = (|a\rangle \otimes |c\rangle)(\langle b| \otimes \langle d|) = |ac\rangle\langle bd|$.

THEOREM 1.6 Let *A* be an $m \times m$ matrix and *B* be a $p \times p$ matrix. Let *A* have the eigenvalues $\lambda_1, \ldots, \lambda_m$ with the corresponding eigenvectors $|u_1\rangle, \ldots, |u_m\rangle$ and let *B* have the eigenvalues μ_1, \ldots, μ_p with the corresponding eigenvectors $|v_1\rangle, \ldots, |v_p\rangle$. Then $A \otimes B$ has mp eigenvalues $\{\lambda_j \mu_k\}$ with the corresponding eigenvectors $\{|u_jv_k\rangle\}.$

Proof. We show that $|u_j v_k\rangle$ is an eigenvector. In fact,

$$
(A \otimes B)(|u_j v_k\rangle) = (A|u_j\rangle) \otimes (B|v_k\rangle) = (\lambda_j |u_j\rangle) \otimes (\mu_k |v_k\rangle)
$$

= $\lambda_j \mu_k (|u_j v_k\rangle)$.

Therefore, the eigenvalue is $\lambda_j \mu_k$ with the corresponding eigenvector $|u_j v_k\rangle$. Since there are *mp* eigenvectors, the vectors $|u_j v_k\rangle$ exhaust all of them.

EXERCISE 1.22 Let *A* and *B* be as above. Show that $A \otimes I_p + I_m \otimes B$ has the eigenvalues $\{\lambda_j + \mu_k\}$ with the corresponding eigenvectors $\{|u_jv_k\rangle\},\$ where I_p is the $p \times p$ unit matrix.

Quantum Mechanics **called the probability and probability** Q

- **1. The state** of a physical system is represented by a normalized vector |ψ> of a suitable Hilbert space. measurement produces one outcome as the produces one outcome and the probability of obtaining of obtaining of o
The probability of obtaining of μ copies μ suitable filibert space.
- **2. Observables** (like position, momentum, spin…) are represented by suitable Hermitian operators. **2. Observables** (like position, momentum, spin...) are
- 3. The state evolved according to the **Schrödinger equation**

$$
i\hbar \frac{\partial |\psi\rangle}{\partial t}=H|\psi\rangle,
$$

It is a linear equation, and implies the **superposition principle:** the linear combination of two possible states is still a possible state of the system.

Quantum Mechanics

4. In a **measurement**, the only possible outcomes are the **eigenvalues** of the Hermitian operator associated to the observable. The outcomes are **random** and distributed with the **Born rule**

$$
\mathbb{P}[c_i] = |\langle c_i | \psi \rangle|^2
$$

where $|c_i\rangle$ is the eigenstate associated to the eigenvalue c_i and $|\psi\rangle$ is the state of the system at the time of the measurement.

5. After the measurement, the state collapses to the eigenstate associated to the measured observable (**von Neumann collapse**)

$$
|\psi\rangle \qquad \longrightarrow \qquad |a_n\rangle
$$

Comments

In Axiom 1, the phase of the vector may be chosen arbitrarily; $|\psi\rangle$ in fact represents the "ray" $\{e^{i\alpha}|\psi\rangle \,|\alpha \in \mathbb{R}\}$. This is called the ray representation. In other words, we can totally igonore the phase of a vector since it has no observable consequence. Note, however, that the *relative* phase of two different states is meaningful. Although $|\langle \phi | e^{i\alpha} \psi \rangle|^2$ is independent of α , $|\langle \phi | \psi_1 + e^{i\alpha} \psi_2 \rangle|^2$ does depend on α .

Axiom 4 may be formulated in a different but equivalent way as follows. Suppose we would like to measure an observable *a*. Let $A = \sum_i \lambda_i |\lambda_i\rangle\langle\lambda_i|$ be the corresponding operator, where $A|\lambda_i\rangle = \lambda_i |\lambda_i\rangle$. Then the expectation value $\langle A \rangle$ of *a* after measurements with respect to many copies of a state $|\psi\rangle$ is

$$
\langle A \rangle = \langle \psi | A | \psi \rangle. \tag{2.2}
$$

Let us expand $|\psi\rangle$ in terms of $|\lambda_i\rangle$ as $|\psi\rangle = \sum_i c_i |\lambda_i\rangle$ to show the equivalence between two formalisms. According to A 2, the probability of observing λ_i upon measurement of *a* is $|c_i|^2$, and therefore the expectation value after many measurements is $\sum_i \lambda_i |c_i|^2$. If, conversely, Eq. (2.2) is employed, we will obtain the same result since

$$
\langle \psi | A | \psi \rangle = \sum_{i,j} c_j^* c_i \langle \lambda_j | A | \lambda_i \rangle = \sum_{i,j} c_j^* c_i \lambda_i \delta_{ij} = \sum_i \lambda_i |c_i|^2.
$$

This measurement is called the projective measurement. Any particular outcome λ_i will be found with the probability

$$
|c_i|^2 = \langle \psi | P_i | \psi \rangle,\tag{2.3}
$$

where $P_i = |\lambda_i\rangle\langle\lambda_i|$ is the projection operator, and the state immediately after the measurement is $|\lambda_i\rangle$ or equivalently

$$
\frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}},\tag{2.4}
$$

where the overall phase has been ignored.

Comments

Comments where the overall phase has been ignored.

The Schrödinger equation (2.1) in Axiom A 3 is formally solved to yield

$$
|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle, \qquad (2.5)
$$

if the Hamiltonian *H* is time-independent, while

$$
|\psi(t)\rangle = \mathcal{T} \exp\left[-\frac{i}{\hbar} \int_0^t H(t)dt\right] |\psi(0)\rangle \tag{2.6}
$$

if *H* depends on *t*, where *T* is the time-ordering operator defined by

$$
\mathcal{T}[A(t_1)B(t_2)] = \begin{cases} A(t_1)B(t_2), & t_1 > t_2 \\ B(t_2)A(t_1), & t_2 \ge t_1 \end{cases}
$$

for a product of two operators. Generalization to products of more than two operators should be obvious. We write Eqs. (2.5) and (2.6) as $|\psi(t)\rangle = U(t)|\psi(0)\rangle$. The operator $U(t): |\psi(0)\rangle \mapsto |\psi(t)\rangle$, which we call the **time-evolution operator**, is unitary. Unitarity of $U(t)$ guarantees that the norm of $|\psi(t)\rangle$ is conserved:

 $\langle \psi(0)|U^{\dagger}(t)U(t)|\psi(0)\rangle = \langle \psi(0)|\psi(0)\rangle = 1.$

Uncertainty principle "ψ(0)*|U†*(*t*)*U*(*t*)*|*ψ(0)! = "ψ(0)*|*ψ(0)! = 1*.*

EXERCISE 2.1 (Uncertainty Principle)

(1) Let *A* and *B* be Hermitian operators and $|\psi\rangle$ be some quantum state on which *A* and *B* operate. Show that

 $|\langle \psi | [A, B] | \psi \rangle|^2 + |\langle \psi | \{A, B\} | \psi \rangle|^2 = 4 |\langle \psi | AB | \psi \rangle|^2.$

(2) Prove the Cauchy-Schwarz inequality

 $|\langle \psi | AB | \psi \rangle|^2 \le \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle.$

(3) Show that

$$
|\langle \psi | [A, B] | \psi \rangle|^2 \le 4 \langle \psi | A^2 | \psi \rangle \langle \psi | B^2 | \psi \rangle.
$$

(4) Show that

$$
\Delta(A)\Delta(B) \ge \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|, \tag{2.7}
$$

2

where $\Delta(A) \equiv \sqrt{\langle \psi | A^2 | \psi \rangle - \langle \psi | A | \psi \rangle^2}$. (5) Suppose $A = Q$ and $B = P \equiv$ \hbar *i d* $\frac{a}{dQ}$. Deduce from the above arguments that $\Delta(Q)\Delta(P) \geq$ \hbar *.*

Example We now give some examples to clarify the axioms introduced introduced in the previous introduced in the previous \mathcal{L}_max E vample LAQIIIN

EXAMPLE 2.1 Let us consider a time-independent Hamiltonian

$$
H = -\frac{\hbar}{2}\omega\sigma_x.
$$
 (2.8)

Suppose the system is in the eigenstate of σ_z with the eigenvalue $+1$ at time $t = 0$;

$$
|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

The wave function $|\psi(t)\rangle$ (*t* > 0) is then found from Eq. (2.5) to be

$$
|\psi(t)\rangle = \exp\left(i\frac{\omega}{2}\sigma_x t\right)|\psi(0)\rangle.
$$
 (2.9)

The matrix exponential function in this equation is evaluated with the help of Eq. (1.44) and we find

$$
|\psi(t)\rangle = \begin{pmatrix} \cos \omega t/2 & i \sin \omega t/2 \\ i \sin \omega t/2 & \cos \omega t/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos \omega t/2 \\ i \sin \omega t/2 \end{pmatrix}.
$$
 (2.10)

Suppose we measure the observable σ_z . Note that $|\psi(t)\rangle$ is expanded in terms of the eigenvectors of σ_z as

$$
|\psi(t)\rangle = \cos\frac{\omega}{2}t|\sigma_z = +1\rangle + i\sin\frac{\omega}{2}t|\sigma_z = -1\rangle.
$$

The state oscillates among the two eigenstates. Why? What should happen to not have the oscillation? What are the probabilities of outcomes of measurements?

Example direction at *t* = 0. Then the spin starts precession around the *x*-axis, and the *z*-component of the spin oscillates sinusoidally as is shown above.

Next let us take the initial state

$$
|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
$$

which is an eigenvector of σ_x (and hence the Hamiltonian) with the eigenvalue +1. We find $|\psi(t)\rangle$ in this case as

$$
|\psi(t)\rangle = \begin{pmatrix} \cos \omega t/2 & i \sin \omega t/2 \\ i \sin \omega t/2 & \cos \omega t/2 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{e^{i\omega t/2}}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$
 (2.11)

Therefore the state remains in its initial state at an arbitrary $t > 0$. This is an expected result since the system at $t = 0$ is an eigenstate of the Hamiltonian.

Exercise Therefore the state remains in its initial state at an arbitrary *t >* 0. This is an

EXERCISE 2.2 Let us consider a Hamiltonian

$$
H = -\frac{\hbar}{2}\omega\sigma_y. \tag{2.12}
$$

Suppose the initial state of the system is

$$
|\psi(0)\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
$$
 (2.13)

(1) Find the wave function $|\psi(t)\rangle$ at later time $t > 0$.

(2) Find the probability for the system to have the outcome $+1$ upon measurement of σ_z at $t > 0$.

 (3) Find the probability for the system to have the outcome $+1$ upon measurement of σ_x at $t > 0$.

Exercise: generalization LACIUISC. SCHCLAIIZAUUIT

Now let us formulate Example 2.1 and Exercise 2.2 in the most general form. Consider a Hamiltonian

$$
H = -\frac{\hbar}{2}\omega \hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma},\qquad(2.14)
$$

where $\hat{\boldsymbol{n}}$ is a unit vector in \mathbb{R}^3 . The time-evolution operator is readily obtained, by making use of the result of Proposition 1.2, as

$$
U(t) = \exp(-iHt/\hbar) = \cos\frac{\omega}{2}t I + i(\hat{\boldsymbol{n}} \cdot \boldsymbol{\sigma})\sin\frac{\omega}{2}t.
$$
 (2.15)

Suppose the initial state is

$$
|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
$$

for example. Then we find

$$
|\psi(t)\rangle = U(t)|\psi(0)\rangle = \begin{pmatrix} \cos(\omega t/2) + in_z \sin(\omega t/2) \\ i(n_x + in_y) \sin(\omega t/2) \end{pmatrix}.
$$
 (2.16)

The reader should verify that $|\psi(t)\rangle$ is normalized at any instant of time $t > 0$.

Bipartite systems

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State

A system composed of two separate components is called bipartite. Then the system as a whole lives in a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, whose general vector is written as

$$
|\psi\rangle = \sum_{i,j} c_{ij} |e_{1,i}\rangle \otimes |e_{2,j}\rangle, \qquad (2.29)
$$

where $\{|e_{a,i}\rangle\}$ $(a = 1, 2)$ is an orthonormal basis in \mathcal{H}_a and $\sum_{i,j} |c_{ij}|^2 = 1$. A state $|\psi\rangle \in \mathcal{H}$ written as a tensor product of two vectors as $|\psi\rangle =$ $|\psi_1\rangle \otimes |\psi_2\rangle$, $(|\psi_a\rangle \in \mathcal{H}_a)$ is called a separable state or a tensor product state. A separable state admits a classical interpretation such as "The first system is in the state $|\psi_1\rangle$, while the second system is in $|\psi_2\rangle$." It is clear that the set of separable states has dimension $\dim \mathcal{H}_1 + \dim \mathcal{H}_2$. Note however that the total space H has different dimensions since we find, by counting the number of coefficients in (2.29), that $\dim \mathcal{H} = \dim \mathcal{H}_1 \dim \mathcal{H}_2$. This number is considerably larger than the dimension of the sparable states when dim \mathcal{H}_a ($a = 1, 2$) are large. What are the missing states then?

Bipartite systems

solution. Therefore the state *|*ψ" is not separable. Such non-separable states are called entangled in quantum theory [9]. The fact

```
\dimH<sub>1</sub>\dimH<sub>2</sub> \gg \dimH<sub>1</sub> + \dimH<sub>2</sub>
```
tells us that most states in a Hilbert space of a bipartite system are entangled when the constituent Hilbert spaces are higher dimensional. These entangled states refuse classical descriptions. Entanglement will be used extensively as a powerful computational resource in quantum information processing and quantum computation.

|
|esson Entanglement is deeply related to quantum nonlocality, the most *i*=1 [√]*si|f*1*,i*! ⊗ *[|]f*2*,i*! with ! *i* fascinating lesson of quantum theory

Schmidt decomposition Suppose a bipartite state (2.29) is given. We are interested in when the schmidt decomposition is separable and when entangled. The schmidt decomposition is given by the Schmidt Schmi C_0 by C_1 \downarrow $\$ SCITITURU MACOMPOSITION real numbers while all the off-diagonal elements vanish. Now *|*ψ! of Eq. (2.29)

PROPOSITION 2.1 Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be the Hilbert space of a bipartite system. Then a vector $|\psi\rangle \in \mathcal{H}$ admits the **Schmidt decomposition** pp oposition

$$
|\psi\rangle = \sum_{i=1}^{r} \sqrt{s_i} |f_{1,i}\rangle \otimes |f_{2,i}\rangle \text{ with } \sum_{i} s_i = 1,
$$
 (2.31)

where $s_i > 0$ are called the **Schmidt coefficients** and $\{|f_{a,i}\rangle\}$ is an orthonormal set of \mathcal{H}_a . The number $r \in \mathbb{N}$ is called the **Schmidt number** of $|\psi\rangle$. where $s_i > 0$ are called the **Schmid**

panded as in Eq. (2.29). Note that the coefficients *cij* form a dim*H*¹ ×dim*H*² The proof will be done in Introduction to Quantum Informati function (2.31) is obtained by replacing the positive number *dⁱ* by *dⁱ* = √*si*. The proof will be done in Introduction to Quantum Information Theory

 $\begin{bmatrix} 1 & f_0 \end{bmatrix}$ in the from if and only if its Schmidt number *r* is 1. *jl|e*1*,i*! ⊗ *|e*2*,j*!*.* It follows from the above proposition that a bipartite state $|\psi\rangle$ is separable

Multipartite systems *[|]f*2*,*1! ⁼ ! *j*=1 *V* ∗ *^j*1*|e*2*,j* ! =

Generalization to a system with more components, i.e., a multipartite system, should be obvious. A system composed of *N* components has a Hilbert space

$$
\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_N, \qquad (2.32)
$$

where \mathcal{H}_a is the Hilbert space to which the *ath* component belongs. Classification of entanglement in a multipartite system is far from obvious, and an analogue of the Schmidt decompostion is not known to date for $N \geq 3.*$