Advanced Quantum Mechanics

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Quantum Integral Transform

DEFINITION 6.1 (**Discrete Integral Transform**) Let $n \in \mathbb{N}$ and $S_n = \{0, 1, \ldots, 2^n - 1\}$ be a set of integers. Consider a map

$$K : S_n \times S_n \to \mathbb{C}. \tag{6.1}$$

For any function $f : S_n \to \mathbb{C}$, its **discrete integral transform** (DIT) $\tilde{f} : S_n \to \mathbb{C}$ with the **kernel** K is defined as:

$$\tilde{f}(y) = \sum_{x=0}^{2^n-1} K(y, x) f(x). \tag{6.2}$$

The transformation $f \to \tilde{f}$ is also called the discrete integral transform.

We define $N \equiv 2^n$ to simplify our notations. The kernel K is expressed as a matrix,

$$K = \begin{pmatrix} K(0,0) & \ldots & K(0, N-1) \\ K(1,0) & \ldots & K(1, N-1) \\ \ldots & \ldots & \ldots \\ K(N-1,0) & \ldots & K(N-1, N-1), \end{pmatrix} \tag{6.3}$$

Quantum Integral Transform

PROPOSITION 6.1 Suppose the kernel K is unitary: $K^\dagger = K^{-1}$. Then the inverse transform $\tilde{f} \to f$ of a DIT exists and is given by

$$f(x) = \sum_{y=0}^{N-1} K^\dagger(x, y)\tilde{f}(y). \tag{6.4}$$

Proof. By substituting Eq. (6.2) into Eq. (6.4), we prove

$$\begin{aligned}\sum_{y=0}^{N-1} K^\dagger(x, y)\tilde{f}(y) &= \sum_{y=0}^{N-1} K^\dagger(x, y)\left[\sum_{z=0}^{N-1} K(y, z)f(z)\right] \\ &= \sum_{z=0}^{N-1}\left[\sum_{y=0}^{N-1} K^\dagger(x, y)K(y, z)\right] f(z) \\ &= \sum_{z=0}^{N-1} \delta_{xz} f(z) = f(x).\end{aligned}$$

∎

Quantum Integral Transform

Now we make the connection with quantum computing

Let U be an $N \times N$ unitary matrix which acts on the n-qubit space $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$. Let $\{|x\rangle = |x_{n-1}, x_{n-2} \ldots, x_0\rangle\}$ $(x_k \in \{0,1\})$ be the standard binary basis of $\mathcal{H}$, where $x = x_{n-1}2^{n-1} + x_{n-2}2^{n-2} + \ldots + x_0 2^0$. Then

$$U|x\rangle = \sum_{y=0}^{N-1} |y\rangle\langle y|U|x\rangle = \sum_{y=0}^{N-1} U(y,x)|y\rangle. \tag{6.5}$$

The complex number $U(x,y) = \langle x|U|y\rangle$ is the (x,y)-component of U in this basis.

Quantum Integral Transform

PROPOSITION 6.2 Let U be a unitary transformation, acting on $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$. Suppose U acts on a basis vector $|x\rangle$ as

$$U|x\rangle = \sum_{y=0}^{N-1} K(y,x)|y\rangle. \tag{6.6}$$

Then U *computes** the DIT $\tilde{f}(y) = \sum_{x=0}^{N-1} K(y,x)f(x)$ for any $y \in S_n$, in the sense that

$$U\left[\sum_{x=0}^{N-1} f(x)|x\rangle\right] = \sum_{y=0}^{N-1} \tilde{f}(y)|y\rangle. \tag{6.7}$$

Here $|x\rangle$ and $|y\rangle$ are basis vectors of $\mathcal{H}$.

Proof. In fact,

$$\begin{aligned} U\left[\sum_{x=0}^{N-1} f(x)|x\rangle\right] &= \sum_{x=0}^{N-1} f(x)U|x\rangle \\ &= \sum_{x=0}^{N-1} f(x)\left[\sum_{y=0}^{N-1} K(y,x)|y\rangle\right] = \sum_{y=0}^{N-1}\left[\sum_{x=0}^{N-1} K(y,x)f(x)\right]|y\rangle \\ &= \sum_{y=0}^{N-1} \tilde{f}(y)|y\rangle. \end{aligned} \tag{6.8}$$

∎

The unitary matrix U implementing a discrete integral transform as in Eq. (6.7) is called the **quantum integral transform (QIT).**

Quantum Integral Transform

We will introduce three types of QIT:

1. Quantum Fourier Transform (QFT)
2. Walsh Hadamard Transform (which we already saw)
3. Selective Phase Rotation Transform

Quantum Fourier Transform

Fourier transform. Let ω_n be the Nth primitive root of 1;

$$\omega_n = e^{2\pi i/N}, \tag{6.10}$$

where $N = 2^n$ as before. The complex number ω_n defines a kernel K by

$$K(x, y) = \frac{1}{\sqrt{N}}\omega_n^{-xy}. \tag{6.11}$$

The discrete integral transform with the kernel K,

$$\tilde{f}(y) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \omega_n^{-xy} f(x), \tag{6.12}$$

The inverse DFT is given by

$$f(x) = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_n^{xy} \tilde{f}(y).$$

is called the **discrete Fourier transform** (**DFT**).

The kernel K is unitary since

$$(KK^\dagger)(x, y) = \langle x|K \sum_z |z\rangle\langle z|K^\dagger|y\rangle = \sum_z K(x, z)K^\dagger(z, y)$$

$$= \frac{1}{N}\sum_z \omega_n^{-xz}\omega_n^{yz} = \frac{1}{N}\sum_z \omega^{-(x-y)z} = \delta_{xy}.$$

Quantum Fourier Transform

The quantum integral transform defined with this kernel is called the **quantum Fourier transform (QFT).**

$$U_{\text{QFT}}|x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \omega_n^{-xy} |y\rangle \quad \Rightarrow \quad U_{\text{QFT}} = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} \omega_n^{-xy} |y\rangle\langle x|$$

It is important to note that

$$U_{\text{QFT}n}|0\rangle = \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n-1} |y\rangle, \tag{6.16}$$

where $U_{\text{QFT}n}$ is the n-qubit QFT gate. This equality shows that the QFT of $f(x) = \delta_{x0}$ is $\tilde{f}(y) = 1/\sqrt{2^n}$, which is similar to the FT of the Dirac delta function $\delta(x)$. Observe that a single application of $U_{\text{QFT}n}$ on the state $|0\rangle$ has produced the superposition of all the basis vectors of $\mathcal{H}$.

Examples

The kernel for $n = 1$ is

$$K_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & e^{2\pi i/2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{6.13}$$

which is nothing but our familiar Hadamard gate. For $n = 2$, we have $\omega_2 = e^{2\pi i/4} = i$ and

$$K_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega_2^{-1} & \omega_2^{-2} & \omega_2^{-3} \\ 1 & \omega_2^{-2} & \omega_2^{-4} & \omega_2^{-6} \\ 1 & \omega_2^{-3} & \omega_2^{-6} & \omega_2^{-9} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}. \tag{6.14}$$

Circuit implementation of QFT: n = 1

$\underline{n = 1}$

Eq. (6.13) shows that the kernel for $n = 1$ QFT is the Hadamard gate H, whose action on $|x\rangle$, $x \in \{0, 1\}$, is concisely written as

$$U_{\mathrm{H}}|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x}|1\rangle) = \frac{1}{\sqrt{2}}\sum_{y=0}^{1}(-1)^{xy}|y\rangle. \qquad (6.24)$$

In fact, this is the defining equation for $n = 1$ QFT as

$$U_{\mathrm{QFT1}}|x\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{1}\omega_{1}^{-xy}|y\rangle = \frac{1}{\sqrt{2}}\sum_{y=0}^{1}(-1)^{xy}|y\rangle. \qquad (6.25)$$

Circuit implementation of QFT: n = 2

$\underline{n = 2}$

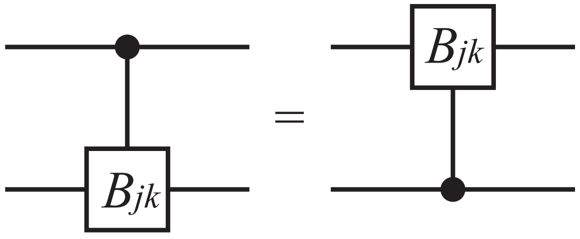
This case is considerably more complicated than the case $n = 1$. It also gives important insights into implementing QFT with $n \geq 3$. Let us introduce an important gate, the **controlled-$\boldsymbol{B_{jk}}$** gate. The B_{jk} gate is defined by the matrix

$$B_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta_{jk}} \end{pmatrix}, \quad \theta_{jk} = \frac{2\pi}{2^{k-j+1}}, \tag{6.26}$$

where $j, k \in \{0, 1, 2, \ldots\}$ and $k \geq j$.

LEMMA 6.1 The controlled-B_{jk} gate U_{jk} in Fig. 6.1 (a) acts on $|x\rangle|y\rangle$, $x, y \in \{0, 1\}$, as

$$U_{jk}|x, y\rangle = e^{-i\theta_{jk}xy}|x, y\rangle = \exp\left(-\frac{2\pi i}{2^{k-j+1}}xy\right)|x, y\rangle. \tag{6.27}$$



(a) (b)

Circuit implementation of QFT: n = 2

Proof. The controlled-B_{jk} gate is written as

$$U_{jk} = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes B_{jk}, \tag{6.28}$$

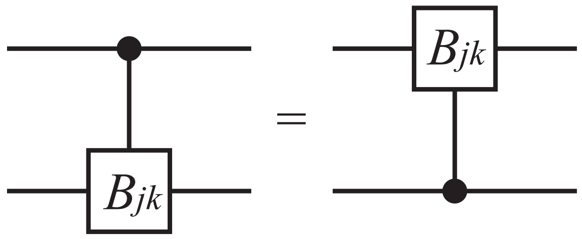
and its action on $|x, y\rangle$ is

$$\begin{aligned} U_{jk}|x,y\rangle &= |0\rangle\langle 0|x\rangle \otimes |y\rangle + |1\rangle\langle 1|x\rangle \otimes B_{jk}|y\rangle \\ &= \begin{cases} |x\rangle \otimes |y\rangle & x = 0 \\ |x\rangle \otimes B_{jk}|y\rangle & x = 1. \end{cases} \end{aligned} \tag{6.29}$$

Moreover, when $x = 1$ we have

$$B_{jk}|y\rangle = \begin{cases} |y\rangle & y = 0 \\ e^{-i\theta_{jk}}|y\rangle & y = 1. \end{cases} \tag{6.30}$$

Thus the action of U_{jk} on $|y\rangle$ is trivial if $xy = 0$ and nontrivial if and only if $x = y = 1$. These results may be summarized as Eq. (6.27). ∎



(a) (b)

The action of the controlled-B_{jk} gate on a basis vector $|x\rangle|y\rangle$ is determined by the combination xy and not by x and y independently. Therefore the controlled-B_{jk} gate and the "inverted" controlled-B_{jk} gate are equivalent; see Fig. 6.1.

Circuit implementation of QFT: n = 2

Equation (6.6) in Proposition 6.2 states that our task is to find a unitary matrix U_{QFT2} such that

$$U_{\mathrm{QFT2}}|x\rangle = \frac{1}{2}\sum_{y=0}^{3} \omega_2^{-xy}|y\rangle. \tag{6.32}$$

Let us write x and y in the binary form as $x = 2x_1 + x_0$ and $y = 2y_1 + y_0$, respectively. The action of U_{QFT2} on $|x\rangle$ is

$$\begin{aligned}
U_{\mathrm{QFT2}}|x_1x_0\rangle &= \frac{1}{2}\sum_{y=0}^{3} e^{-2\pi i xy/2^2}|y\rangle = \frac{1}{2}\sum_{y_0,y_1=0}^{1} e^{-2\pi i x(2y_1+y_0)/2^2}|y_1y_0\rangle \\
&= \frac{1}{2}\sum_{y_1} e^{-2\pi i x y_1/2}|y_1\rangle \otimes \sum_{y_0} e^{-2\pi i x y_0/2^2}|y_0\rangle \\
&= \frac{1}{2}\left(|0\rangle + e^{-2\pi i x/2}|1\rangle\right) \otimes \left(|0\rangle + e^{-2\pi i x/2^2}|1\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle + e^{-2\pi i(2x_1+x_0)/2}|1\rangle\right) \otimes \left(|0\rangle + e^{-2\pi i(2x_1+x_0)/2^2}|1\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle + e^{-\pi i x_0}|1\rangle\right) \otimes \left(|0\rangle + e^{-\pi i x_1} e^{-i(\pi/2)x_0}|1\rangle\right) \\
&= \frac{1}{2}\left(|0\rangle + (-1)^{x_0}|1\rangle\right) \otimes B_{12}^{x_0}\left(|0\rangle + (-1)^{x_1}|1\rangle\right),
\end{aligned} \tag{6.33}$$

$$B_{12} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta_{12}} \end{pmatrix}, \quad \theta_{12} = \frac{2\pi}{2^{2-1+1}} = \frac{\pi}{2}$$

Then

$$B_{12} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/2} \end{pmatrix}$$

Note that $B^{x_0}_{12}$ is the controlled-B gate with the control bit x_0 and the target bit x_1; $B^0_{12} = I$ while $B^1_{12} = B_{12}$. Note also that, in spite of its tensor product looking appearance, the last line of Eq. (6.33) is entangled due to this conditional operation.

Circuit implementation of QFT: n = 2

$$U_{\mathrm{QFT2}}|x_1x_0\rangle = \frac{1}{\sqrt{2^2}}\left(|0\rangle + (-1)^{x_0}|1\rangle\right) \otimes B_{12}^{x_0}\left(|0\rangle + (-1)^{x_1}|1\rangle\right)$$

Equation (6.33) suggests that the $n = 2$ QFT are implemented with the Hadamard and the U_{12} gates. Before writing down the quantum circuit realizing Eq. (6.33), we should note that the first qubit has a power $(-1)^{x_0}$, while the second one has $(-1)^{x_1}$, when the input state is $|x_1x_0\rangle$. If we naively applied the Hadamard gate to the second qubit, we would obtain

$$(I \otimes U_{\mathrm{H}})|x_1x_0\rangle = |x_1\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_0}|1\rangle).$$

These facts suggest that we need to swap the first and second qubits at the beginning of the implementation

Circuit implementation of QFT: n = 2

$$U_{\rm QFT2}|x_1x_0\rangle = \frac{1}{\sqrt{2^2}}\left(|0\rangle + (-1)^{x_0}|1\rangle\right) \otimes B_{12}^{x_0}\left(|0\rangle + (-1)^{x_1}|1\rangle\right)$$
$$= (U_{\rm H} \otimes I)U_{12}(I \otimes U_{\rm H})|x_0, x_1\rangle$$
$$= (U_{\rm H} \otimes I)U_{12}(I \otimes U_{\rm H})U_{\rm SWAP}|x_1x_0\rangle. \qquad (6.34)$$

PROPOSITION 6.3 The $n = 2$ QFT gate is implemented as

$$U_{\rm QFT2} = (U_{\rm H} \otimes I)U_{12}(I \otimes U_{\rm H})U_{\rm SWAP} \qquad (6.35)$$

(see Fig. 6.2).

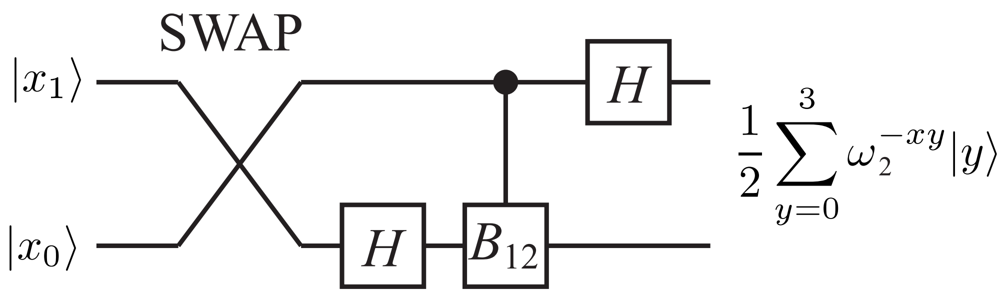


FIGURE 6.2
Implementation of the $n = 2$ QFT, $U_{\rm QFT2}$.

Circuit implementation of QFT: n = 3

$$
\begin{aligned}
&U_{\mathrm{QFT3}}|x_2x_1x_0\rangle \\
&= \frac{1}{\sqrt{2^3}}(|0\rangle + e^{-2\pi i x_0/2}|1\rangle) \otimes (|0\rangle + e^{-2\pi i(x_1/2+x_0/2^2)}|1\rangle) \\
&\quad \otimes(|0\rangle + e^{-2\pi i(x_2/2+x_1/2^2+x_0/2^3)}|1\rangle) \\
&= \frac{1}{\sqrt{2^3}}(|0\rangle + (-1)^{x_0}|1\rangle) \otimes B_{01}^{x_0}(|0\rangle + (-1)^{x_1}|1\rangle) \\
&\quad \otimes B_{02}^{x_0} B_{12}^{x_1}(|0\rangle + (-1)^{x_2}|1\rangle) \\
&= (U_{\mathrm{H}} \otimes I \otimes I)U_{01}(I \otimes U_{\mathrm{H}} \otimes I)U_{02}U_{12}(I \otimes I \otimes U_{\mathrm{H}})|x_0x_1x_2\rangle \\
&= (U_{\mathrm{H}} \otimes I \otimes I)U_{01}(I \otimes U_{\mathrm{H}} \otimes I)U_{02}U_{12}(I \otimes I \otimes U_{\mathrm{H}})P|x_2x_1x_0\rangle, \quad (6.36)
\end{aligned}
$$

$$
\begin{aligned}
|x_1x_0\rangle \to \frac{1}{\sqrt{2^2}} \sum_{y=0}^{2^2-1} e^{-2\pi i x y/2^2}|y\rangle \\
= \frac{1}{\sqrt{2^2}}(|0\rangle + e^{-2\pi i x_0/2}|1\rangle) \otimes (|0\rangle + e^{-2\pi i(x_1/2+x_0/2^2)}|1\rangle).
\end{aligned}
$$

For n= 2

where U_{jk} is the controlled-B_{jk} gate with the control qubit x_j, and the gate P reverses the order of the qubits as $P|x_2x_1x_0\rangle = |x_0x_1x_2\rangle$. For a three-qubit QFT, P is a SWAP gate between the first qubit (x_2) and the third qubit (x_0). Again note here that we should be careful in ordering the gates so that the control bit x_j acts in U_{jk} before it is acted by a Hadamard gate.

Circuit implementation of QFT: n = 3

$$U_{\mathrm{QFT3}} = (U_{\mathrm{H}} \otimes I \otimes I)U_{01}(I \otimes U_{\mathrm{H}} \otimes I)U_{02}U_{12}(I \otimes I \otimes U_{\mathrm{H}})P. \qquad (6.38)$$

Equation (6.38) readily leads us to the quantum circuit in Fig. 6.3.

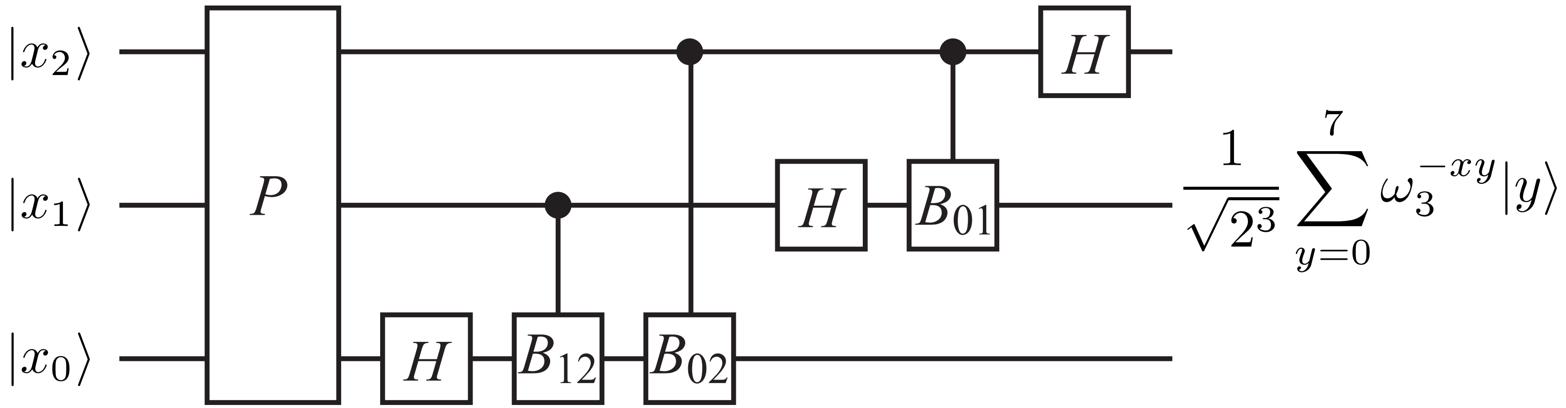


FIGURE 6.3
Implementation of the $n = 3$ QFT.

Exercise

EXERCISE 6.5 Let $x = 2^2 x_2 + 2x_1 + x_0$ and $y = 2^2 y_2 + 2y_1 + y_0$.
(1) Write down the RHS of

$$U_{\mathrm{QFT3}}|x_2 x_1 x_0\rangle = \frac{1}{\sqrt{2^3}} \sum_{y=0}^{2^3-1} e^{-2\pi i x y/2^3} |y\rangle \tag{6.37}$$

explicitly in terms of x_i and y_i.
(2) Show that the RHS of Eq. (6.37) agrees with the first line of the RHS of Eq. (6.36).

Since Eq. (6.36) is true for any $|x_2 x_1 x_0\rangle$, we have found

$$U_{\mathrm{QFT3}} = (U_{\mathrm{H}} \otimes I \otimes I) U_{01} (I \otimes U_{\mathrm{H}} \otimes I) U_{02} U_{12} (I \otimes I \otimes U_{\mathrm{H}}) P. \tag{6.38}$$

Circuit implementation of QFT: n general

Now the generalization of the present construction to $n \geq 4$ should be easy. The equation that generalizes Eq. (6.36) is

$$
\begin{aligned}
&U_{\mathrm{QFT}n}|x_{n-1}\dots x_1 x_0\rangle \\
&= \frac{1}{\sqrt{N}}(|0\rangle + e^{-2\pi i x_0/2}|1\rangle) \otimes (|0\rangle + e^{-2\pi i(x_1/2 + x_0/2^2)}|1\rangle) \\
&\quad \otimes(|0\rangle + e^{-2\pi i(x_2/2 + x_1/2^2 + x_0/2^3)}|1\rangle) \otimes \dots \\
&\quad \dots \otimes (|0\rangle + e^{-2\pi i(x_{n-1}/2 + x_{n-2}/2^2 + \dots x_1/2^{n-1} + x_0/2^n)}|1\rangle) \\
&= (U_{\mathrm{H}} \otimes I \otimes \dots \otimes I) U_{01} (I \otimes U_{\mathrm{H}} \otimes I \otimes \dots \otimes I) U_{02} U_{12} \\
&\quad \times (I \otimes I \otimes U_{\mathrm{H}} \otimes \dots \otimes I) \dots \\
&\quad \times U_{0,n-1} U_{1,n-1} \dots U_{n-2,n-1} (I \otimes \dots \otimes I \otimes U_{\mathrm{H}}) |x_0 x_1 \dots x_{n-1}\rangle \\
&= (U_{\mathrm{H}} \otimes I \otimes \dots \otimes I) U_{01} (I \otimes U_{\mathrm{H}} \otimes I \otimes \dots \otimes I) U_{02} U_{12} \\
&\quad \times (I \otimes I \otimes U_{\mathrm{H}} \otimes \dots \otimes I) \dots U_{0,n-1} U_{1,n-1} \dots U_{n-2,n-1} \\
&\quad \times (I \otimes \dots \otimes I \otimes U_{\mathrm{H}}) P |x_{n-1} \dots x_1 x_0\rangle, \qquad (6.39)
\end{aligned}
$$

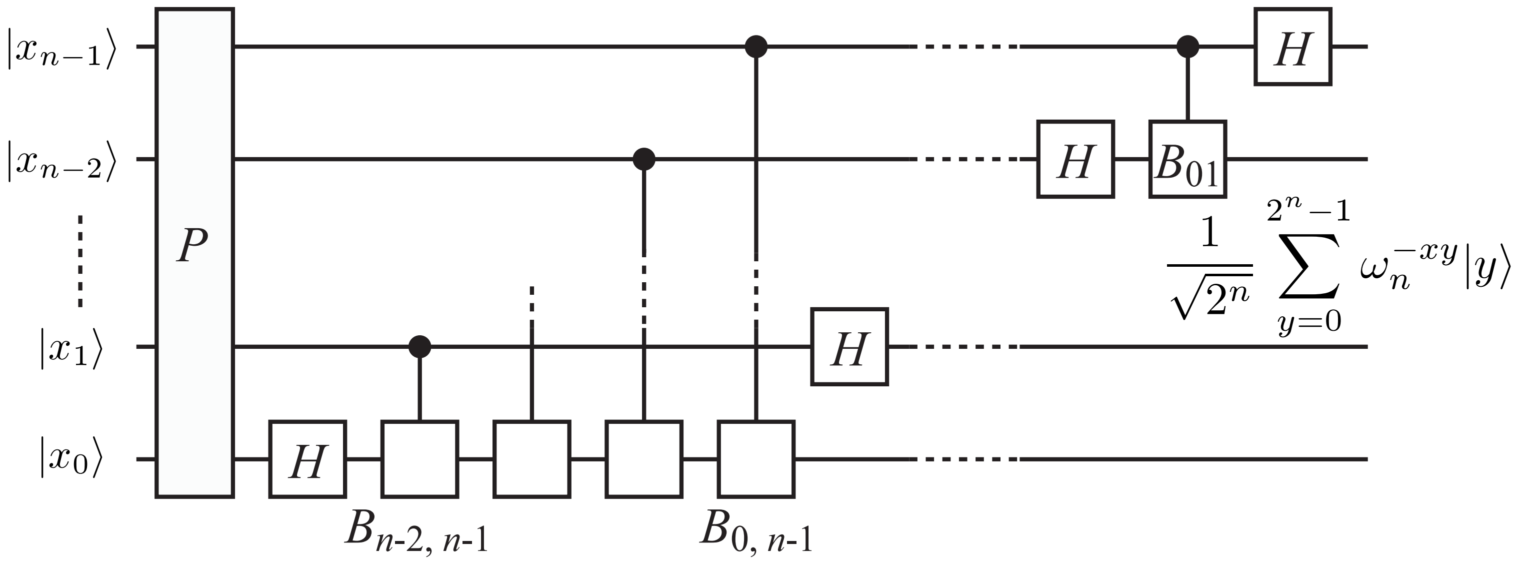
where P reverses the order of x_k as $P|x_{n-1}\dots x_1 x_0\rangle = |x_0 x_1 \dots x_{n-1}\rangle$.

Circuit implementation of QFT: n general

We finally find the following decompostion of $U_{\mathrm{QFT}n}$:

$$\begin{aligned} U_{\mathrm{QFT}n} = & (U_{\mathrm{H}} \otimes I \otimes \ldots \otimes I) U_{01} (I \otimes U_{\mathrm{H}} \otimes I \otimes \ldots \otimes I) U_{02} U_{12} \\ & \times (I \otimes I \otimes U_{\mathrm{H}} \otimes \ldots \otimes I) \ldots \\ & \times U_{0,n-1} U_{1,n-1} \ldots U_{n-2,n-1} (I \otimes \ldots \otimes I \otimes U_{\mathrm{H}}) P. \end{aligned} \quad (6.40)$$

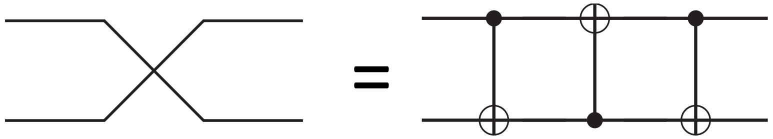
A quantum circuit which implements $U_{\mathrm{QFT}n}$ is found from Eq. (6.40) as in Fig. 6.4. It may be proved, by induction, for example, that the circuit in

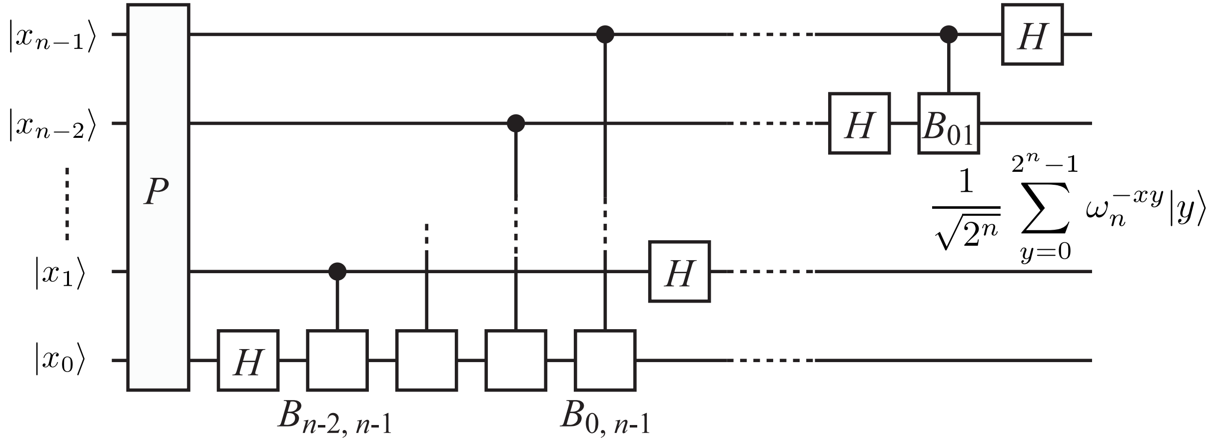


Circuit implementation of QFT: n general

PROPOSITION 6.4 The n-qubit QFT may be constructed with $\Theta(n^2)$ elementary gates.

Proof. The n-qubit QFT is made of a P gate, n Hadamard gates and $(n-1)+(n-2)+\ldots+2+1 = n(n-1)/2$ controlled-B_{jk} gates (see Fig. 6.4). It has been shown in §4.2.3 that it requres three CNOT gates to construct a SWAP gate. Furthermore, a P gate for n qubits requires $\lfloor n/2 \rfloor$ SWAP gates,† assuming that there exists a SWAP gate for any pair of qubits. Thus a P gate requires $3 \times \lfloor n/2 \rfloor = \Theta(n)$ elementary gates. Proposition 4.1 states that a controlled-B_{ij} gate is constructed with at most six elementary gates. Thus it has been proved that the n-qubit QFT is made of $\Theta(n^2)$ elementary gates.





Walsh Hadamard Transform

We have already encountered the Walsh-Hadamard transform in §4.2.2 and §5.2. Let $x, y \in S_n = \{0, 1, \ldots, N-1\}$ with binary expressions $x_{n-1}x_{n-2}\ldots x_0$ and $y_{n-1}y_{n-2}\ldots y_0$, where $N = 2^n$. The Walsh-Hadamard transform, written in the form of Eq. (5.7), shows that it is a quantum integral transform with a kernel $W_n : S_n \times S_n \to \mathbb{C}$ defined by

$$W_n(x, y) = \frac{1}{\sqrt{N}}(-1)^{x \cdot y} \quad (x, y \in S_n), \tag{6.41}$$

where $x \cdot y = x_{n-1}y_{n-1} \oplus x_{n-2}y_{n-2} \oplus \ldots \oplus x_0 y_0$. This kernel defines a discrete integral transform

$$\tilde{f}(y) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{x \cdot y} f(x). \tag{6.42}$$

$$U_{\mathrm{H}}|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x|1\rangle) = \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{xy}|y\rangle,$$

$$\begin{aligned} W_n|x\rangle &= (U_{\mathrm{H}}|x_{n-1}\rangle)(U_{\mathrm{H}}|x_{n-2}\rangle)\ldots(U_{\mathrm{H}}|x_0\rangle) \\ &= \frac{1}{\sqrt{2^n}} \sum_{y_{n-1}, y_{n-2}, \ldots, y_0 \in \{0,1\}} (-1)^{x_{n-1}y_{n-1} + x_{n-2}y_{n-2} + \ldots + x_0 y_0} \\ &\quad \times |y_{n-1}y_{n-2}\ldots y_0\rangle \\ &= \frac{1}{\sqrt{2^n}} \sum_{y=0}^{2^n - 1} (-1)^{x \cdot y}|y\rangle, \end{aligned} \tag{5.7}$$

Selective Phase Rotation Transform

DEFINITION 6.2 (**Selective Phase Rotation Transform**) Let us define a kernel

$$K_n(x, y) = e^{i\theta_x}\delta_{xy}, \quad \forall x, y \in S_n, \tag{6.43}$$

where $\theta_x \in \mathbb{R}$. The discrete integral transform

$$\tilde{f}(y) = \sum_{x=0}^{N-1} K(x, y)f(x) = \sum_{x=0}^{N-1} e^{i\theta_x}\delta_{xy}f(x) = e^{i\theta_y}f(y) \tag{6.44}$$

with the kernel K_n is called the **selective phase rotation transform**.

EXERCISE 6.7 Show that K_n defined above is unitary. Write down the inverse transformation K_n^{-1}.

Selective Phase Rotation Transform

The matrix representations for K_1 and K_2 are

$$K_1 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}, \quad K_2 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}.$$

Selective Phase Rotation Transform

The implementation of K_n is achieved with the universal set of gates as follows. Take $n = 2$, for example. The kernel K_2 has been given above. This is decomposed as a product of two two-level unitary matrices as

$$K_2 = A_0 A_1, \tag{6.45}$$

where

$$A_0 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}. \tag{6.46}$$

Note that

$$A_0 = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes I, \quad U_0 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix},$$

$$A_1 = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U_1, \quad U_1 = \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{i\theta_3} \end{pmatrix}.$$

Selective Phase Rotation Transform

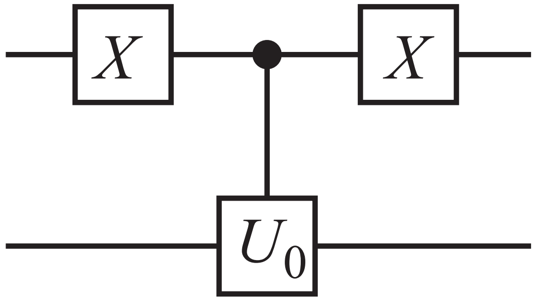
Thus A_1 is realized as an ordinary controlled-U_1 gate while the control bit is negated in A_0. Then what we have to do for A_0 is to negate the control bit first and then to apply ordinary controlled-U_0 gate and finally to negate the control bit back to its input state. In summary, A_0 is implemented as in Fig. 6.5. In fact, it can be readily verified that the gate in Fig. 6.5 is written as

$$A_0 = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes I,$$

$$A_1 = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U_1,$$

$$\begin{aligned}&(X \otimes I)(|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U_0)(X \otimes I)\\&= X|0\rangle\langle 0|X \otimes I + X|1\rangle\langle 1|X \otimes U_0 = |1\rangle\langle 1| \otimes I + |0\rangle\langle 0| \otimes U_0 = A_0.\end{aligned}$$

Thus these gates are implemented with the set of universal gates. In fact, the order of A_i does not matter since $[A_0, A_1] = 0$.



Back on Grover's search algorithm

We need to prove that the D gate used to perform the quantum search can be implemented efficiently. We now show that

$$D = W_n R_0 W_n, \tag{7.6}$$

where W_n is the Walsh-Hadamard transform,

$$W_n(x, y) = \frac{1}{\sqrt{N}}(-1)^{x \cdot y}, \quad (x, y \in S_n) \tag{7.7}$$

and R_0 is the selective phase rotation transform defined by

$$R_0(x, y) = e^{i\pi(1-\delta_{x0})}\delta_{xy} = (-1)^{1-\delta_{x0}}\delta_{xy}. \tag{7.8}$$

Back on Grover's search algorithm

Proof

$$\langle x|D|y\rangle = \langle x|\left[-I + 2|\varphi_o\rangle\langle\varphi_0|\right]|y\rangle = -\delta_{xy} + \frac{2}{N}$$

$$|\varphi_0\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$$

$$\langle x|W_n R_0 W_n|y\rangle = \sum_{u,v}\langle x|W_n|u\rangle\langle u|R_0|v\rangle\langle v|W_n|y\rangle = \frac{1}{N}\sum_{u,v}(-1)^{x\cdot u}(-1)^{1-\delta_{u0}}\delta_{uv}(-1)^{v\cdot y}.$$

$$= \frac{1}{N}\sum_{u}(-1)^{x\cdot u}(-1)^{y\cdot u}(-1)^{1-\delta_{u0}}$$

$$= \frac{1}{N}\left[1 - \sum_{u\neq 0}(-1)^{x\cdot u}(-1)^{y\cdot u}\right]$$

Back on Grover's search algorithm

$$\frac{1}{N}\left[1-\sum_{u\neq 0}(-1)^{x\cdot u}(-1)^{y\cdot u}\right]=A$$

x = y: $$A=\frac{1}{N}\left[1-\sum_{u\neq 0}\right]=\frac{1}{N}\left[1-(N-1)\right]=-1+\frac{2}{N}$$

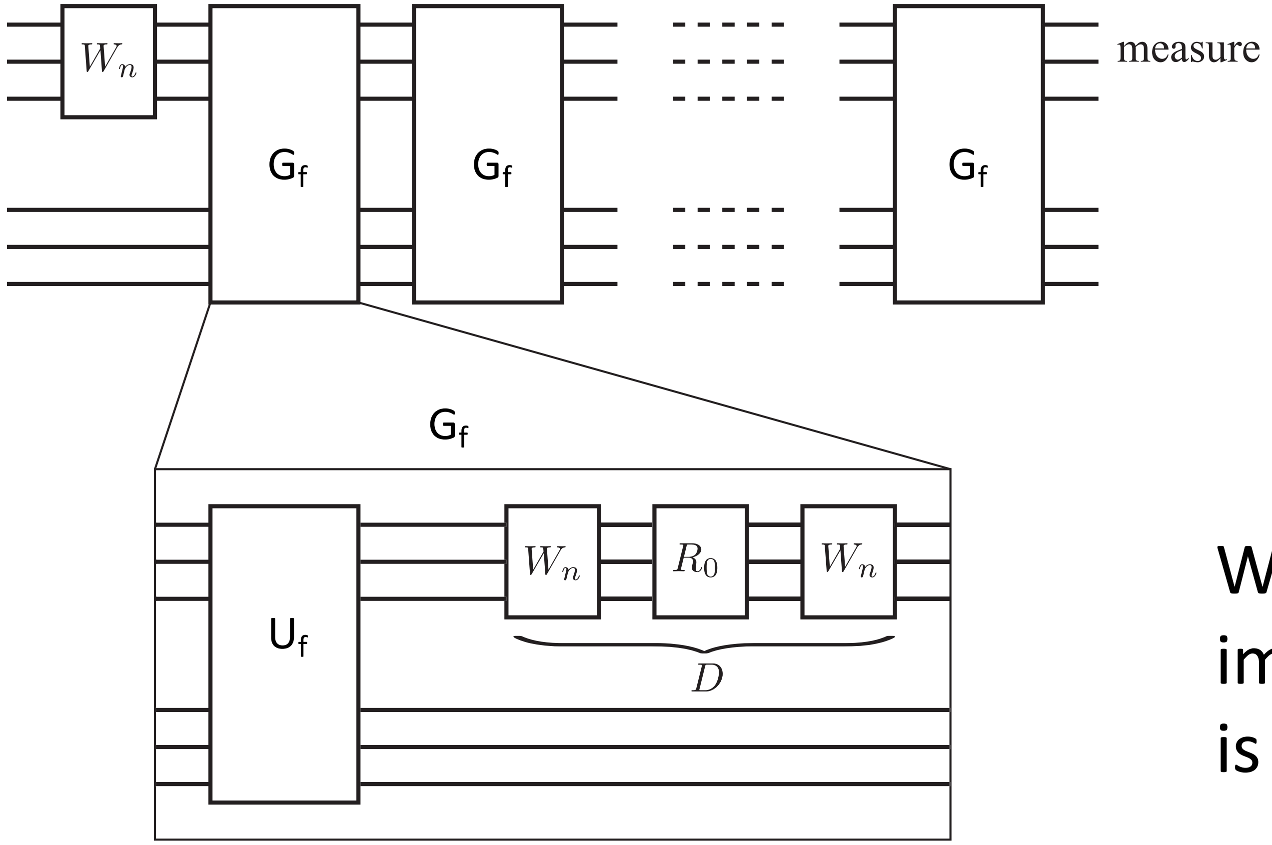
x ≠ y. As discussed in relation to the Deutsch-Jozsa algorithm

$$\sum_{u=0}^{N-1}(-1)^{x\cdot u}=0\ \rightarrow\ \sum_{u\neq 0}^{N-1}(-1)^{x\cdot u}=-1$$

Therefore: $$A=\frac{1}{N}\left[1-(-1)\right]=\frac{2}{N}$$

Back on Grover's search algorithm

Therefore the D gate can be implemented efficiently. The overall circuit is



We are not interested on how to implement the oracle U_f since this is supposed to be given

Shor’s factorization algorithm

Shor's algorithm is a polynomial-time quantum computer algorithm for integer factorization. It solves the following problem: Given an integer N, find its prime factors. It was invented in 1994 by Peter Shor.

Shor's algorithm consists of two parts:

1. A reduction, which can be done on a classical computer, of the factoring problem to the problem of **order-finding**.
2. A quantum algorithm to solve the order-finding problem.

The first part can be done easily. We will see the second part.

Order finding – the problem

Number to factorize

Define $f_N : \mathbb{N} \to \mathbb{N}$ by $a \mapsto m^a \bmod N$. Find the smallest $P \in \mathbb{N}$, such that $m^P \equiv 1 \bmod N$. The number P is called the **order** or **period**. It is known that this takes exponentially large steps in any classical algorithm, but it takes only polynomial steps in Shor's algorithm. A quantum computer is required only in this step, and the rest may be executed in polynomial steps even with a classical computer.

Order finding – the quantum solution

Our quantum computer has two n-qubit registers which we call $|\mathrm{REG1}\rangle$ and $|\mathrm{REG2}\rangle$:

$$|\mathrm{REG1}\rangle|\mathrm{REG2}\rangle = |a\rangle|b\rangle = |a_{n-1}\dots a_1 a_0\rangle|b_{n-1}\dots b_1 b_0\rangle, \tag{8.7}$$

where decimal numbers $a, b \in S_n$ are expressed in binary numbers in the RHS;

$$a = \sum_{j=0}^{n-1} a_j 2^j, \;\; b = \sum_{j=0}^{n-1} b_j 2^j.$$

Step 0. Set the registers to the initial state

$$|\psi_0\rangle = |\mathrm{REG1}\rangle|\mathrm{REG2}\rangle = |\underbrace{00\dots0}_{n \text{ qubits}}\rangle|\underbrace{00\dots0}_{n \text{ qubits}}\rangle. \tag{8.9}$$

Order finding – the quantum solution

Step 1. The QFT $\mathcal{F}$ is applied on the first register;

$$|\psi_0\rangle = |0\rangle|0\rangle \stackrel{\mathcal{F}\otimes I}{\mapsto} |\psi_1\rangle = \frac{1}{\sqrt{Q}}\sum_{x=0}^{Q-1}|x\rangle|0\rangle. \quad (8.10)$$

The first register is in a superposition of all the states $|x\rangle$ $(0 \le x \le Q-1)$,

with $Q = 2^n$. Remember that QFT on all |0>'s gives the equal weighted superposition of all computational basis states

Order finding – the quantum solution

Step 2. Let us define a function f :

$$f(x) = m^x \bmod N, \quad x \in S_n = \{0, 1, \ldots, Q-1\} \tag{8.11}$$

Suppose that the unitary gate U_f realizes the action of f on x in such a way that $U_f|x\rangle|0\rangle = |x\rangle|f(x)\rangle$. Apply U_f on the state prepared in step 2.1 to yield

$$U_f|\psi_1\rangle = |\psi_2\rangle \equiv \frac{1}{\sqrt{Q}} \sum_{x=0}^{Q-1} |x\rangle|f(x)\rangle. \tag{8.12}$$

Order finding – the quantum solution

Step 3. Apply QFT on $|\text{REG1}\rangle$ again to yield

$$
\begin{aligned}
|\psi_3\rangle = (\mathcal{F} \otimes I)|\psi_2\rangle &= \frac{1}{Q} \sum_{x=0}^{Q-1} \sum_{y=0}^{Q-1} \omega_n^{-xy} |y\rangle |f(x)\rangle \\
&= \frac{1}{Q} \sum_{y=0}^{Q-1} |y\rangle |\Upsilon(y)\rangle = \frac{1}{Q} \sum_{y=0}^{Q-1} \| |\Upsilon(y)\rangle \| \cdot |y\rangle \frac{|\Upsilon(y)\rangle}{\| |\Upsilon(y)\rangle \|},
\end{aligned}
\tag{8.13}
$$

where

$$
|\Upsilon(y)\rangle = \sum_{x=0}^{Q-1} \omega_n^{-xy} |f(x)\rangle. \tag{8.14}
$$

Order finding – the quantum solution

Step 4. $|\mathrm{REG1}\rangle$ is measured. The result $y \in S_n$ is obtained with the probability

$$\mathrm{Prob}(y) = \frac{\| |\Upsilon(y)\rangle \|^2}{Q^2}, \qquad (8.15)$$

and, at the same time, the state collapses to

$$|y\rangle \frac{|\Upsilon(y)\rangle}{\| |\Upsilon(y)\rangle \|}.$$

The measurement process generates a random variable following a classical probability distribution $\mathcal{S}$ over S_n, in which "symbols" $y \in S_n$ are generated with the probability (8.15).

Step 5. Extract the order P from the measurement outcome.

Order finding – the quantum solution

P is what we want to find

PROPOSITION 8.1 Let $Q = 2^n = Pq + r$, $(0 \le r < P)$, where q and r are uniquely determined non-negative integers. Let $Q_0 = Pq$. Then

$$\text{Prob}(y) = \begin{cases} \dfrac{r\sin^2\left(\frac{\pi Py}{Q}\left(\frac{Q_0}{P}+1\right)\right) + (P-r)\sin^2\left(\frac{\pi Py}{Q}\cdot\frac{Q_0}{P}\right)}{Q^2\sin^2\left(\frac{\pi Py}{Q}\right)} & (Py \not\equiv 0 \bmod Q) \\ \dfrac{r(Q_0+P)^2 + (P-r)Q_0^2}{Q^2P^2} & (Py \equiv 0 \bmod Q). \end{cases}$$

Proof. It is found from the definition that¶

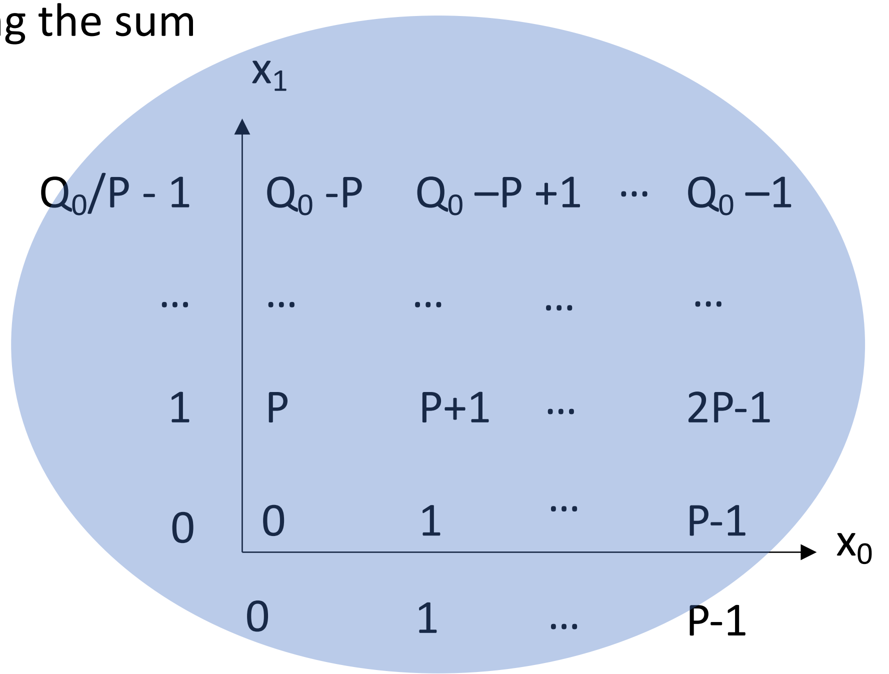
$$|\Upsilon(y)\rangle = \sum_{x=0}^{Q-1} \omega^{-xy}|f(x)\rangle = \sum_{x=0}^{Q_0-1} \omega^{-xy}|f(x)\rangle + \sum_{x=Q_0}^{Q-1} \omega^{-xy}|f(x)\rangle$$

Definition + splitting the sum

$$= \sum_{x_0=0}^{P-1} \sum_{x_1=0}^{Q_0/P-1} \omega^{-(Px_1+x_0)y}|f(Px_1+x_0)\rangle$$

$x = Px_1 + x_0$

$$+ \sum_{x_0=0}^{r-1} \omega^{-[P(Q_0/P)+x_0]y}|f(P(Q_0/P)+x_0)\rangle$$



$$= \sum_{x_0=0}^{P-1} \omega^{-x_0 y} \left(\sum_{x_1=0}^{Q_0/P-1} \omega^{-Px_1 y} \right) |f(x_0)\rangle + \sum_{x_0=0}^{r-1} \omega^{-x_0 y} \omega^{-Py(Q_0/P)} |f(x_0)\rangle$$

$$= \sum_{x_0=0}^{r-1} \omega^{-x_0 y} \sum_{x_1=0}^{Q_0/P-1} \omega^{-Pyx_1}|f(x_0)\rangle$$

$$+ \sum_{x_0=r}^{P-1} \omega^{-x_0 y} \sum_{x_1=0}^{Q_0/P-1} \omega^{-Pyx_1}|f(x_0)\rangle + \sum_{x_0=0}^{r-1} \omega^{-x_0 y} \omega^{-Py(Q_0/P)} |f(x_0)\rangle$$

Splitting the sum

$$= \sum_{x_0=0}^{r-1} \omega^{-x_0 y} \left(\sum_{x_1=0}^{Q_0/P} \omega^{-Pyx_1} \right) |f(x_0)\rangle$$

Merges the two sums

$$+ \sum_{x_0=r}^{P-1} \omega^{-x_0 y} \left(\sum_{x_1=0}^{Q_0/P-1} \omega^{-Pyx_1} \right) |f(x_0)\rangle.$$

So far we have

$$|\Upsilon(y)\rangle = \sum_{x_0=0}^{r-1} \omega^{-x_0 y} \left(\sum_{x_1=0}^{Q_0/P} \omega^{-Pyx_1} \right) |f(x_0)\rangle + \sum_{x_0=r}^{P-1} \omega^{-x_0 y} \left(\sum_{x_1=0}^{Q_0/P-1} \omega^{-Pyx_1} \right) |f(x_0)\rangle.$$

Note that the map $f : a \mapsto m^a \bmod N$ is $1:1$ on $\{0,1,2,\ldots,P-1\}$

This implies that $|f(0)\rangle, |f(1)\rangle, \ldots, |f(P-1)\rangle$ are mutually orthogonal. Accordingly

$$\langle \Upsilon(y)|\Upsilon(y)\rangle = r \left| \sum_{x_1=0}^{Q_0/P} \omega^{-Pyx_1} \right|^2 + (P-r) \left| \sum_{x_1=0}^{Q_0/P-1} \omega^{-Pyx_1} \right|^2.$$

In case $Py \equiv 0 \bmod Q$, we put $Py = aQ,\ a \in \mathbb{N}$ and obtain

$$\omega^{-Pyx_1} = e^{-2\pi i(Py/Q)x_1} = e^{-2\pi i a x_1} = 1.$$

Therefore

$$\langle \Upsilon(y) | \Upsilon(y) \rangle = r \cdot \left(\frac{Q_0}{P} + 1 \right)^2 + (P - r) \left(\frac{Q_0}{P} \right)^2,$$

which leads to the result independent of y,

$$\mathrm{Prob}(y) = \frac{r(Q_0 + P)^2 + (P - r)Q_0^2}{P^2 Q^2} = \frac{r(q+1)^2 + (P - r)q^2}{Q^2}. \qquad (8.16)$$

If $Py \not\equiv 0 \bmod Q$, on the other hand, we obtain

$$\begin{aligned}\langle\Upsilon(y)|\Upsilon(y)\rangle &= r\left|\frac{\omega^{-Py(Q_0/P+1)}-1}{\omega^{-Py}-1}\right|^2 + (P-r)\left|\frac{\omega^{-Py(Q_0/P)}-1}{\omega^{-Py}-1}\right|^2 \\ &= r\left|\frac{e^{-(2\pi i/Q)Py(Q_0/P+1)}-1}{e^{-(2\pi i/Q)Py}-1}\right|^2 + (P-r)\left|\frac{e^{-(2\pi i/Q)Py(Q_0/P)}-1}{e^{-(2\pi i/Q)Py}-1}\right|^2\end{aligned}$$

Here we find from

$$|e^{i\theta}-1|^2 = 2(1-\cos\theta) = 4\sin^2\frac{\theta}{2}$$

that

$$\langle\Upsilon(y)|\Upsilon(y)\rangle = r\frac{\sin^2\frac{\pi}{Q}Py\left(\frac{Q_0}{P}+1\right)}{\sin^2\frac{\pi}{Q}Py} + (P-r)\frac{\sin^2\frac{\pi}{Q}Py\frac{Q_0}{P}}{\sin^2\frac{\pi}{Q}Py}.$$

Therefore, the probability distribution is given by

$$\mathrm{Prob}(y) = \frac{\||\Upsilon(y)\rangle\|^2}{Q^2} = \frac{r\sin^2\left[\frac{\pi}{Q}Py\left(\frac{Q_0}{P}+1\right)\right] + (P-r)\sin^2\left[\frac{\pi}{Q}Py\frac{Q_0}{P}\right]}{Q^2\sin^2\frac{\pi}{Q}Py}, \tag{8.17}$$

which proves the proposition. ∎

COROLLARY 8.1 Suppose $Q/P \in \mathbb{Z}$ (namely $Q_0 = Q$). Then the probability of obtaining a measurement outcome y is → **r = 0**

$$\mathrm{Prob}(y) = \begin{cases} 0 & (Py \not\equiv 0 \bmod Q) \\ \dfrac{1}{P} & (Py \equiv 0 \bmod Q) \end{cases}$$

→ Peaks are repeated at distance q, because we are in the first situation until y = q

Proof. When $Py \not\equiv 0 \bmod Q$, $r = 0$ implies $Q = Pq$. Therefore

$$\mathrm{Prob}(y) = \frac{P \sin^2 \pi y}{Q^2 \sin^2 \frac{\pi y}{q}} = 0.$$

In case $Py \equiv 0 \bmod Q$, we obtain

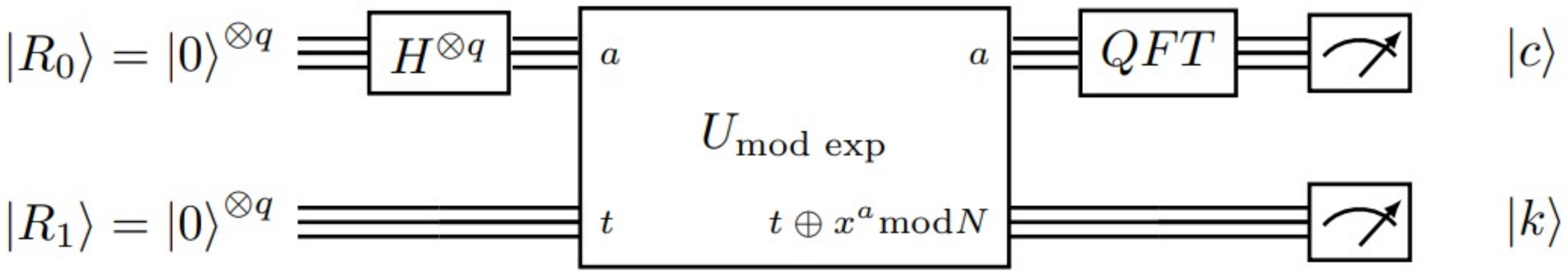
$$\mathrm{Prob}(y) = \frac{PQ^2}{Q^2 P^2} = \frac{1}{P}.$$

∎

Factoring 15 (Credits: Dr. G. Crognaletti)

N = 15.
m = 7.
Quindi: f(x) = 7^x mod 15

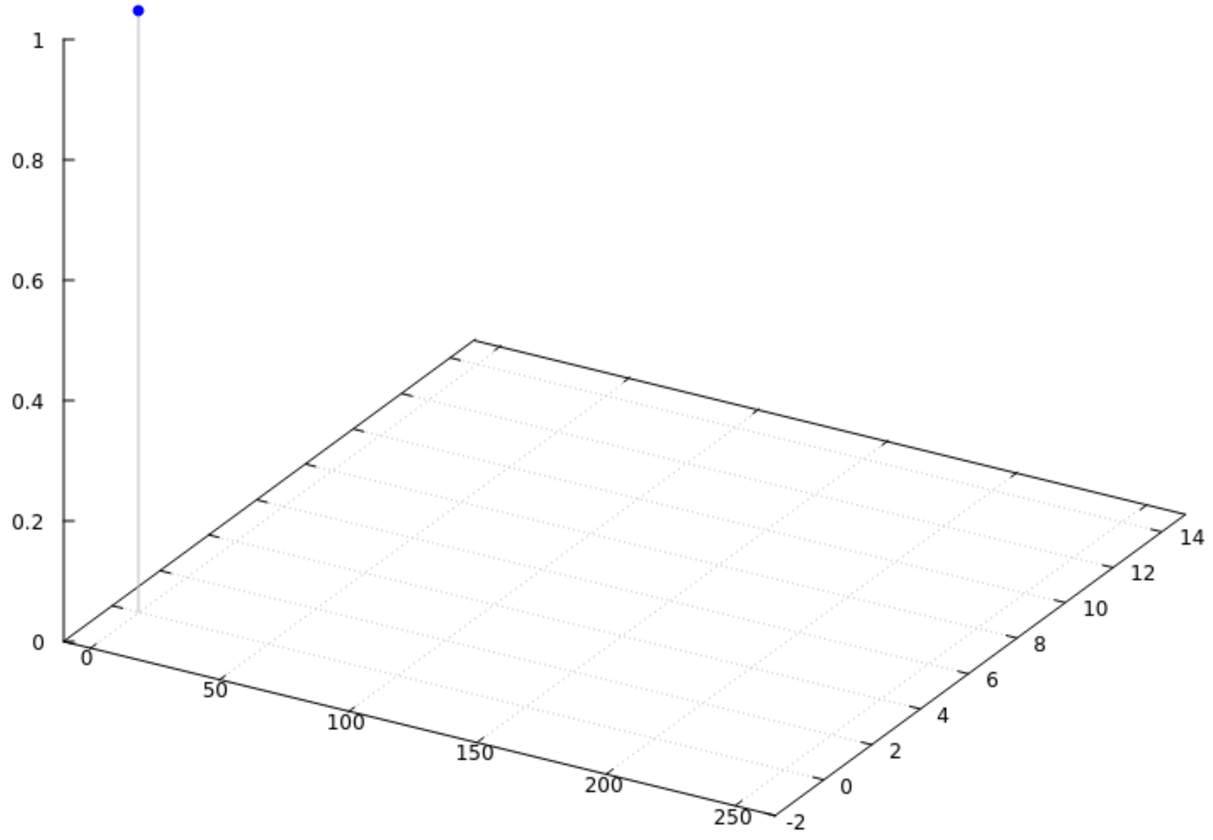
Il numero di qubit n è stabilito da 2 $\log_2(N) < n < 2 \log_2(N)+1$, in questo caso $7.8 < n < 8.8 \Rightarrow n = 8$, necessito di $2^8 = 256$ ampiezze di probabilità.



Factoring 15

Ad ogni stato della macchina è associato un istogramma di questo tipo:

- Il primo asse rappresenta la base computaizionale del primo registro, i cui valori verranno indicati con c.
- Il secondo rappresenta la base computazionale del secondo, limitato ai valori ottenuti nell pratica (in questo caso 13), i cui valori verranno indicati con k.
- L'asse verticale rappresenta la probabilità di misura P(c,k) associata ad ogni elemento della base della coppia di registri. Es: lo stato iniziale

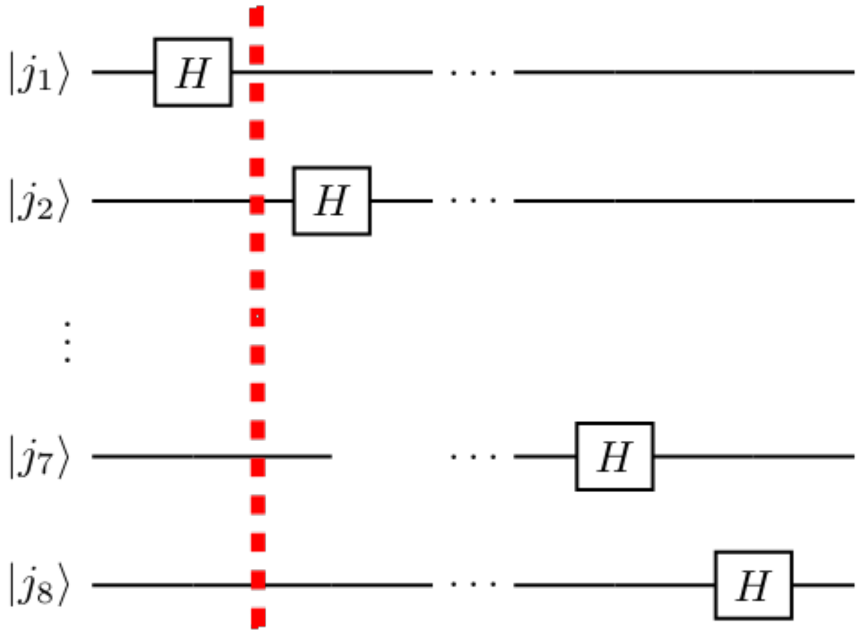


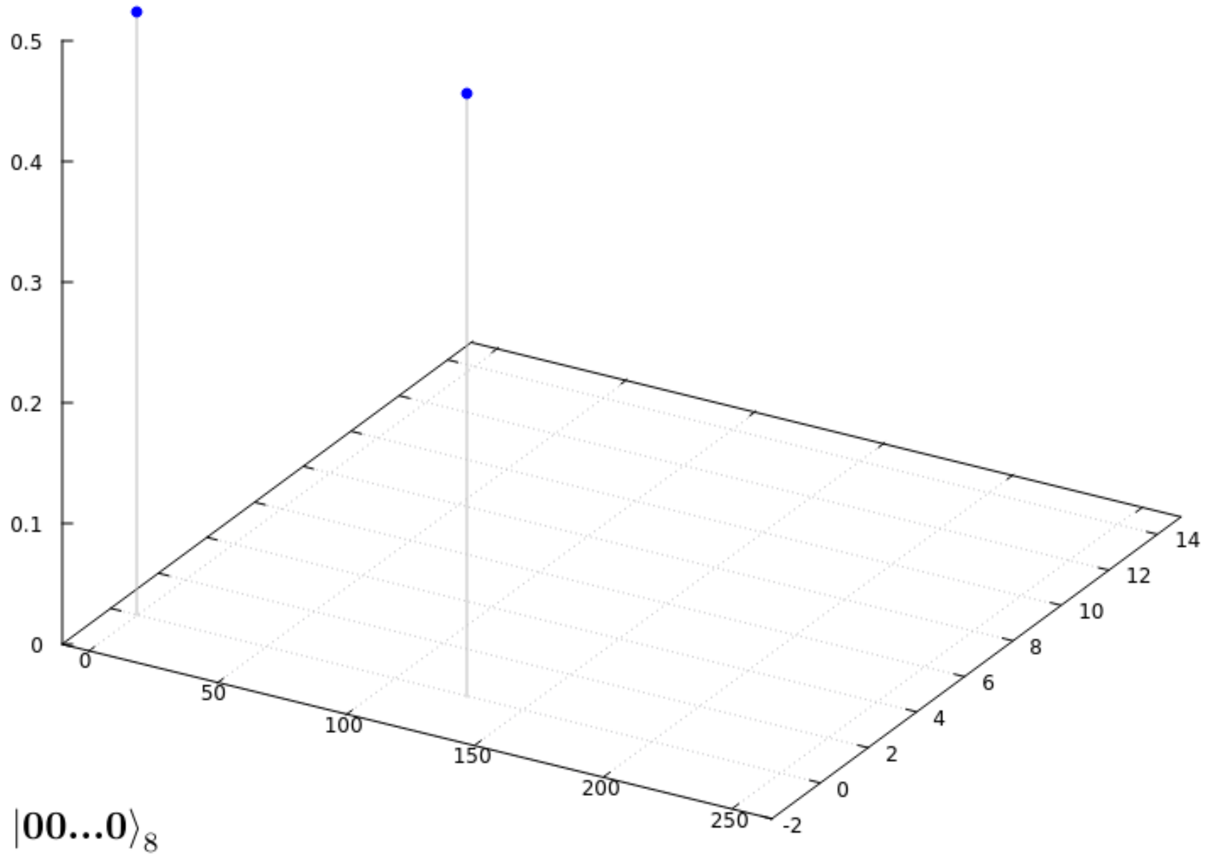
$$|R_0\rangle\,|R_1\rangle = |\mathbf{00...0}\rangle_8\,|\mathbf{00...0}\rangle_8 = |0\rangle\,|0\rangle$$

Factoring 15

1. **Trasformata di Hadamard**

Creo lo stato sovrapposto di tutta la base computazionale. Ciò richiede in totale n operazioni (applicazione di H ad ognuno dei Qubit)



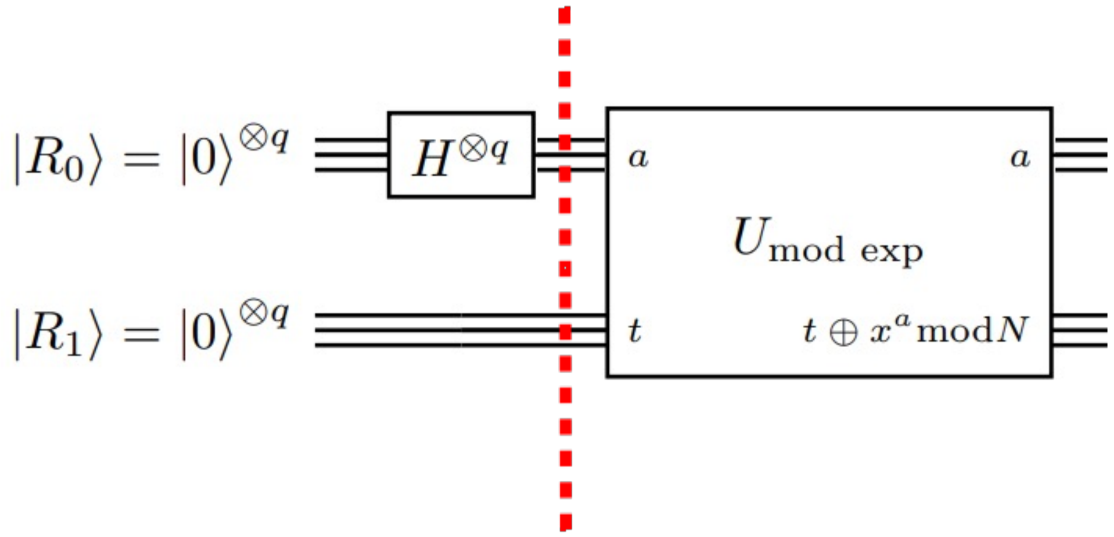


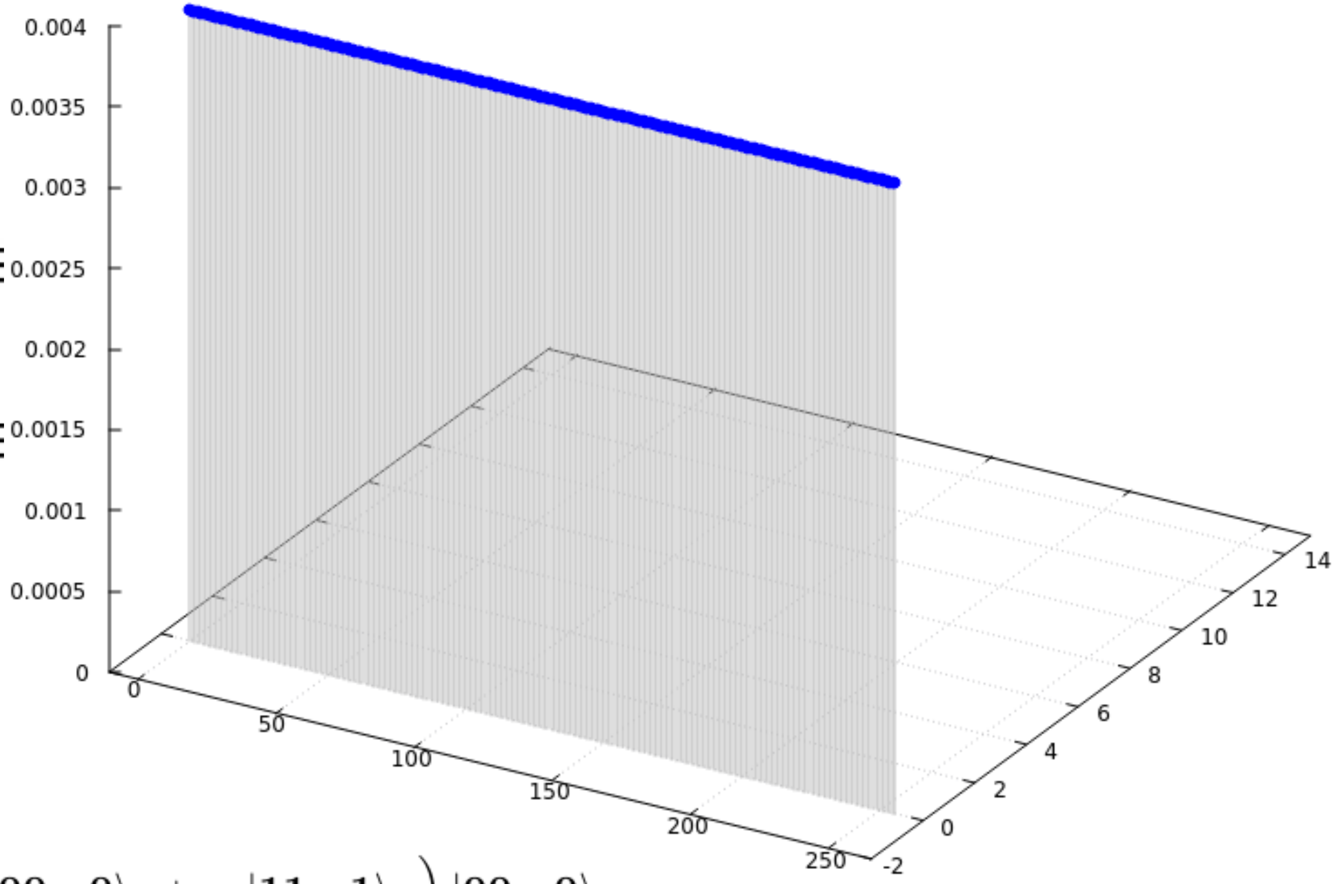
$$|R_0\rangle|R_1\rangle = \left[\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|\mathbf{00...0}\rangle_7\right]|\mathbf{00...0}\rangle_8 = \frac{1}{\sqrt{2}}\left(|\mathbf{00..0}\rangle_8 + |\mathbf{10...0}\rangle_8\right)|\mathbf{00...0}\rangle_8$$

=0 =128

Factoring 15

1. **Trasformata di Hadamard**





$$|R_0\rangle\,|R_1\rangle = \left[\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \dots \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\right] |\mathbf{00...0}\rangle_8 = \frac{1}{\sqrt{256}}\left(|\mathbf{00...0}\rangle_8 + \dots |\mathbf{11...1}\rangle_8\right) |\mathbf{00...0}\rangle_8$$

=0 =255

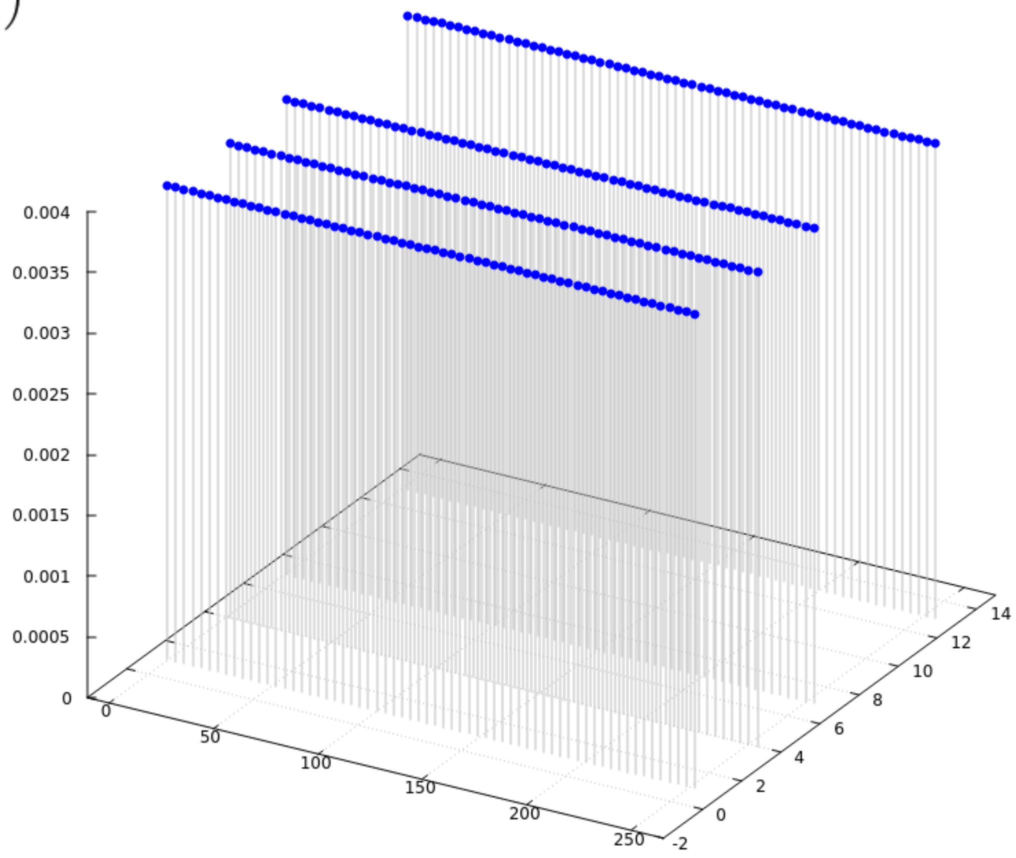
Factoring 15

2. **Applico l'operatore esponenziale modulare U_f**

$$|R_0\rangle\,|R_1\rangle = \frac{1}{\sqrt{256}}\Big(\,|0\rangle\,|1\rangle + |1\rangle\,|7\rangle + |2\rangle\,|4\rangle + |3\rangle\,|13\rangle + |4\rangle\,|1\rangle + |5\rangle\,|7\rangle \ldots |255\rangle\,|13\rangle\,\Big)$$

È uno stato non separabile, descrivibile come sovrapposizione con uguale ampiezza di probabilità di 4 stati separabili

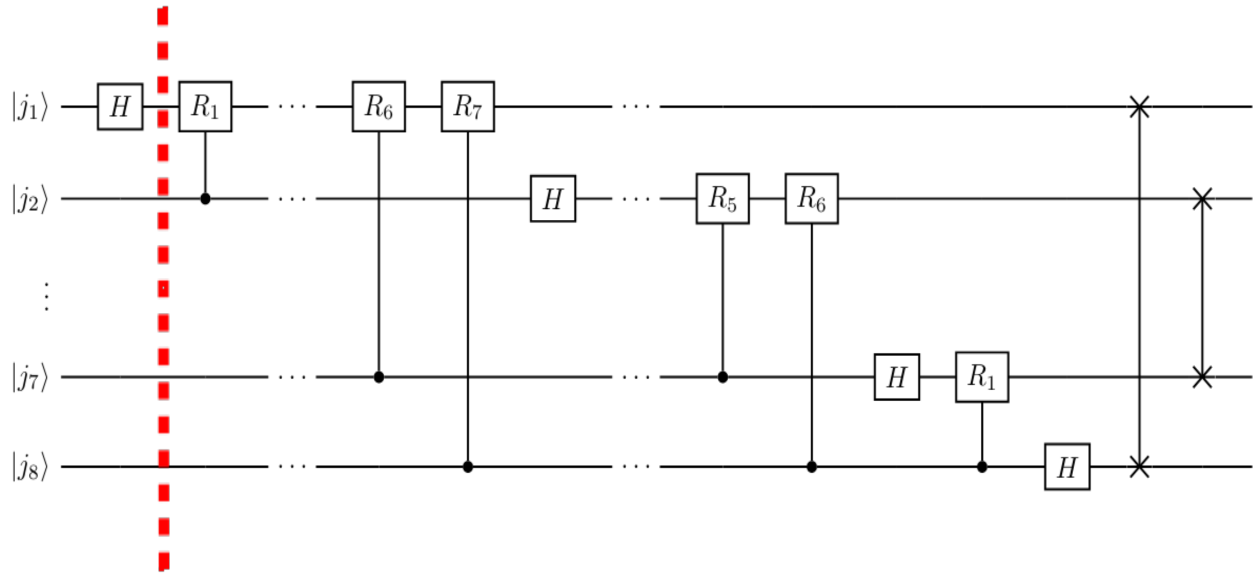
$$|R_0\rangle\,|R_1\rangle = \frac{1}{\sqrt{4}}\left(\frac{|0\rangle + |4\rangle + \ldots + |252\rangle}{\sqrt{64}}\right)|1\rangle + \frac{1}{\sqrt{4}}\left(\frac{|1\rangle + |5\rangle + \ldots + |253\rangle}{\sqrt{64}}\right)|7\rangle + \frac{1}{\sqrt{4}}\left(\frac{|2\rangle + |6\rangle + \ldots + |254\rangle}{\sqrt{64}}\right)|4\rangle + \frac{1}{\sqrt{4}}\left(\frac{|3\rangle + |7\rangle + \ldots + |255\rangle}{\sqrt{64}}\right)|13\rangle$$



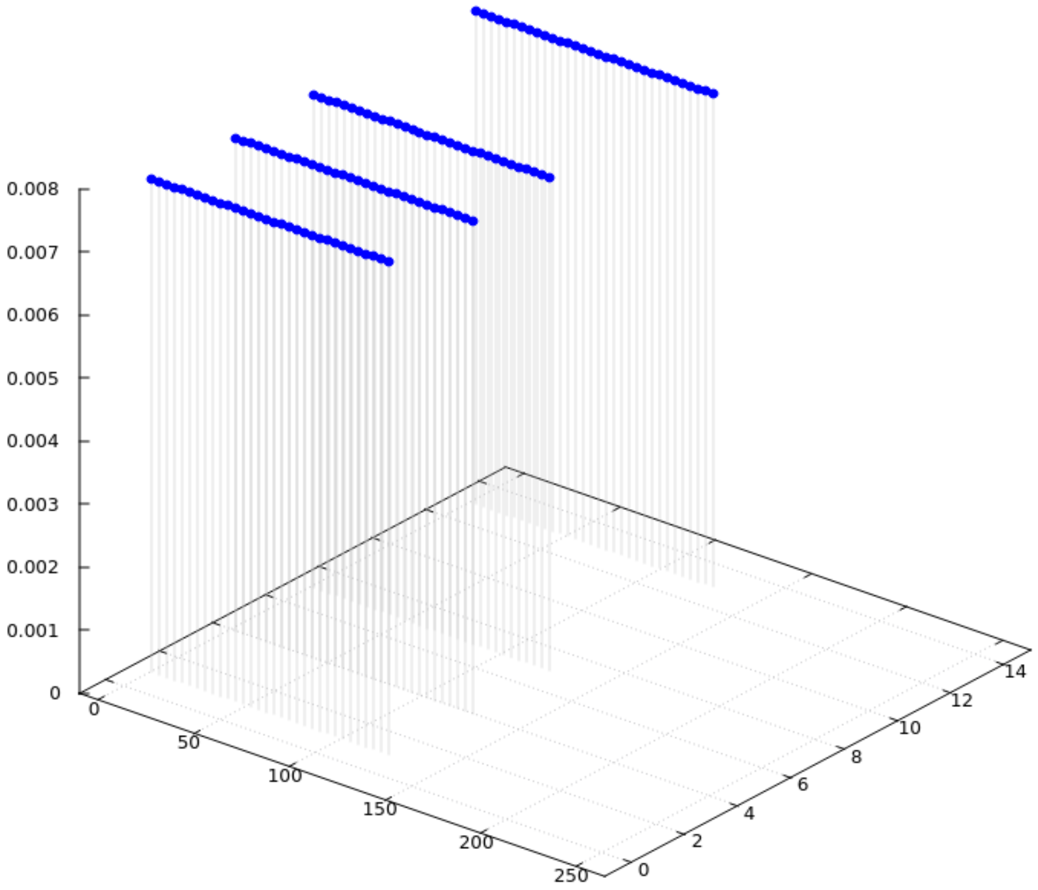
Factoring 15

3. **Applico la Trasformata di Fourier Quantistica**

L'algoritmo utilizzato in questo caso è quello relativo alla QFT esatta, schematizzato dal circuito:

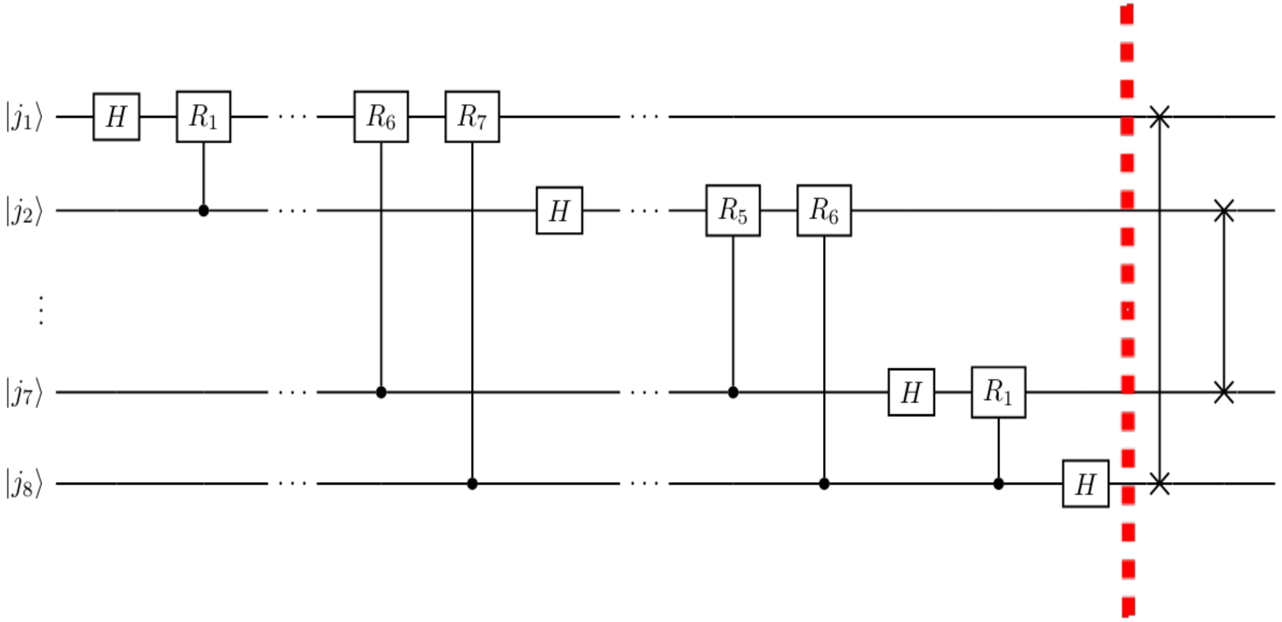


$$|R_0\rangle|R_1\rangle = \frac{1}{\sqrt{4}}\left(\frac{|0\rangle+|4\rangle+...+|124\rangle}{\sqrt{32}}\right)|1\rangle + \frac{1}{\sqrt{4}}\left(\frac{|1\rangle+|5\rangle+...+|125\rangle}{\sqrt{32}}\right)|7\rangle + \frac{1}{\sqrt{4}}\left(\frac{|2\rangle+|6\rangle+...+|126\rangle}{\sqrt{32}}\right)|4\rangle + \frac{1}{\sqrt{4}}\left(\frac{|3\rangle+|7\rangle+...+|127\rangle}{\sqrt{32}}\right)|13\rangle$$

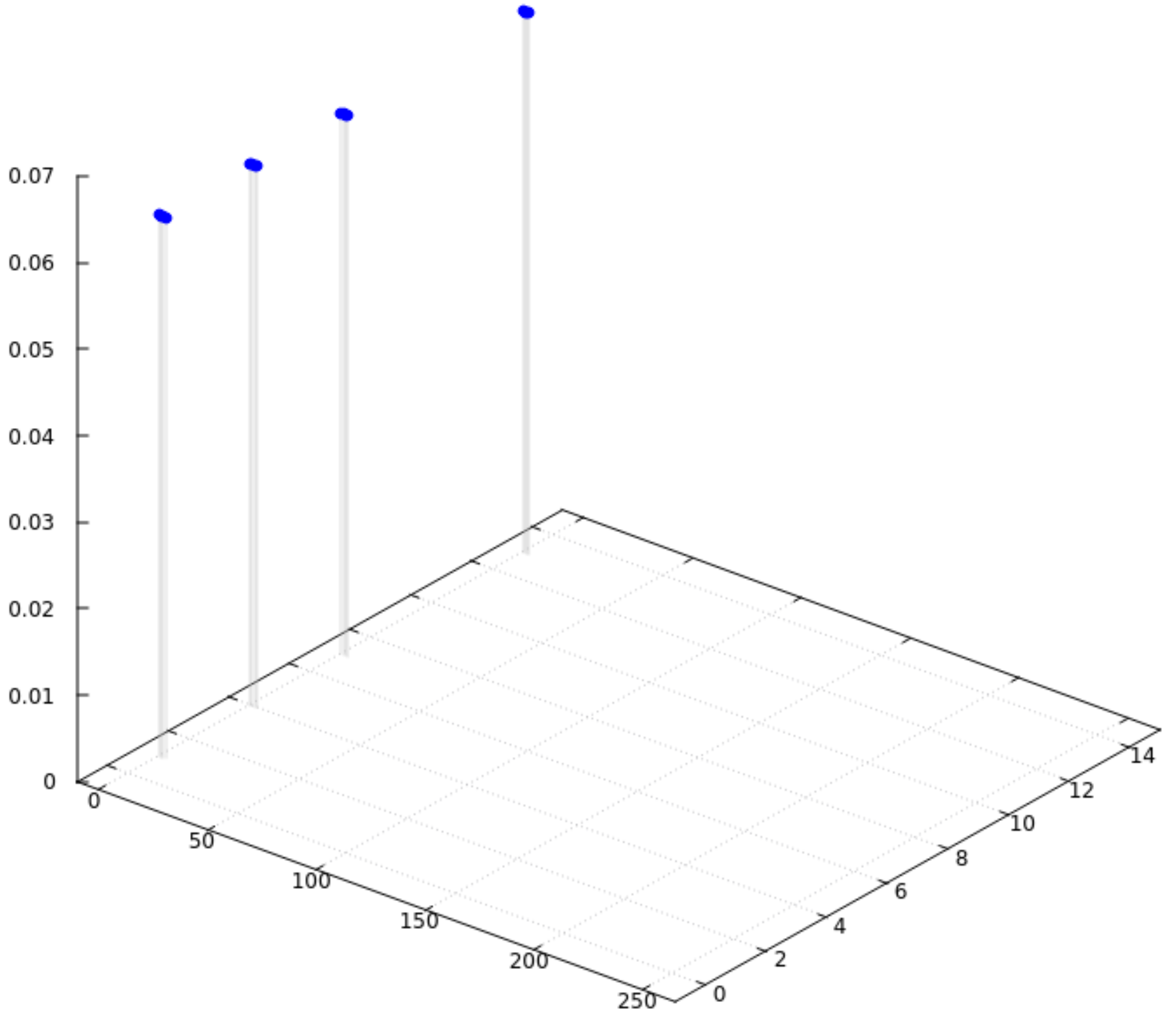


Factoring 15

3. **Applico la Trasformata di Fourier Quantistica**



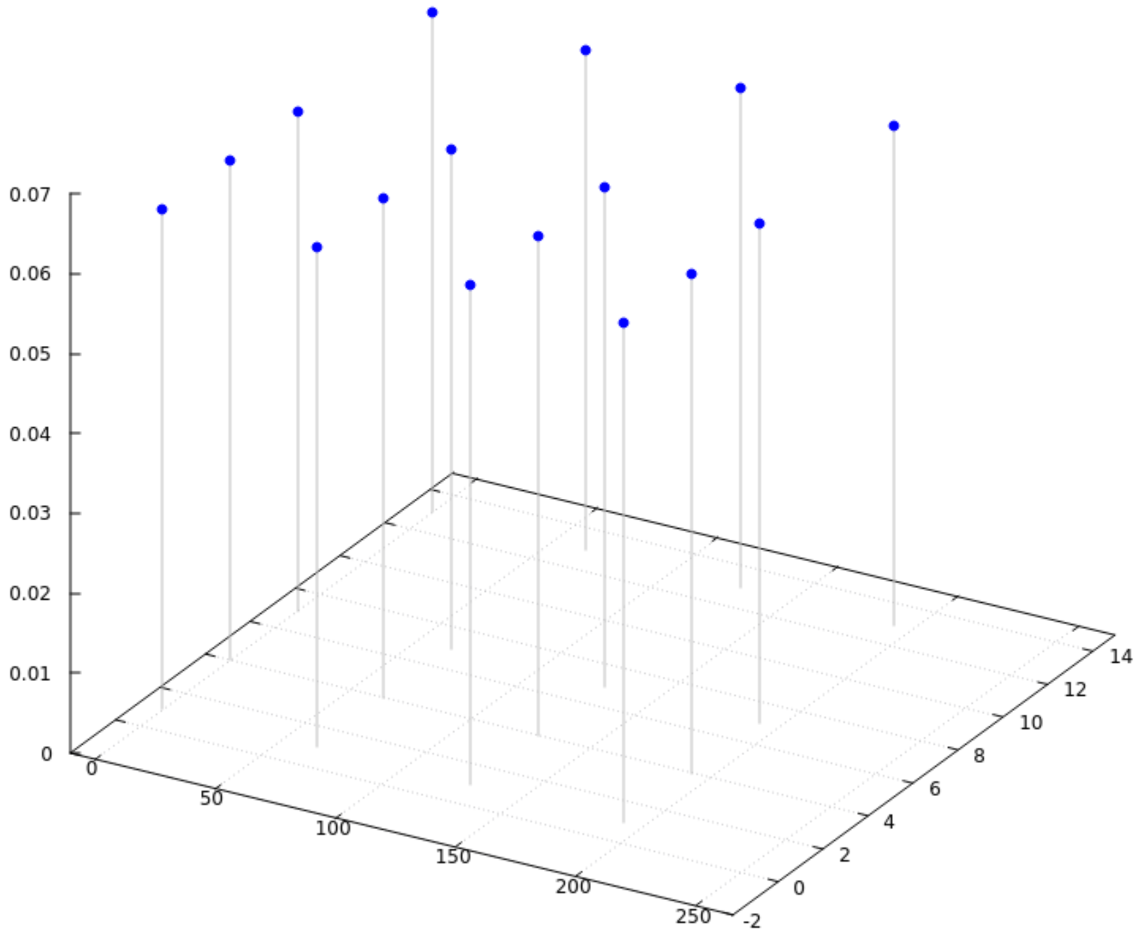
$$|R_0\rangle\,|R_1\rangle = \frac{1}{\sqrt{4}}\left(\frac{|0\rangle+|1\rangle+|2\rangle+|3\rangle}{\sqrt{4}}\right)|1\rangle + \frac{1}{\sqrt{4}}\left(\frac{|0\rangle-|1\rangle+i\,|2\rangle-i\,|3\rangle}{\sqrt{4}}\right)|7\rangle$$
$$+\frac{1}{\sqrt{4}}\left(\frac{|0\rangle+|1\rangle-|2\rangle-|3\rangle}{\sqrt{4}}\right)|4\rangle + \frac{1}{\sqrt{4}}\left(\frac{|0\rangle-|1\rangle-i\,|2\rangle+i\,|3\rangle}{\sqrt{4}}\right)|13\rangle$$



Factoring 15

3. **Applico la Trasformata di Fourier Quantistica**

$$|R_0\rangle|R_1\rangle = \frac{1}{\sqrt{4}}|0\rangle\left(\frac{|1\rangle+|7\rangle+|4\rangle+|13\rangle}{\sqrt{4}}\right) + \frac{1}{\sqrt{4}}|64\rangle\left(\frac{|1\rangle+i|7\rangle-|4\rangle-i|13\rangle}{\sqrt{4}}\right) + \frac{1}{\sqrt{4}}|128\rangle\left(\frac{|1\rangle-|7\rangle+|4\rangle-|13\rangle}{\sqrt{4}}\right) + \frac{1}{\sqrt{4}}|192\rangle\left(\frac{|1\rangle-i|7\rangle-|4\rangle+i|13\rangle}{\sqrt{4}}\right)$$



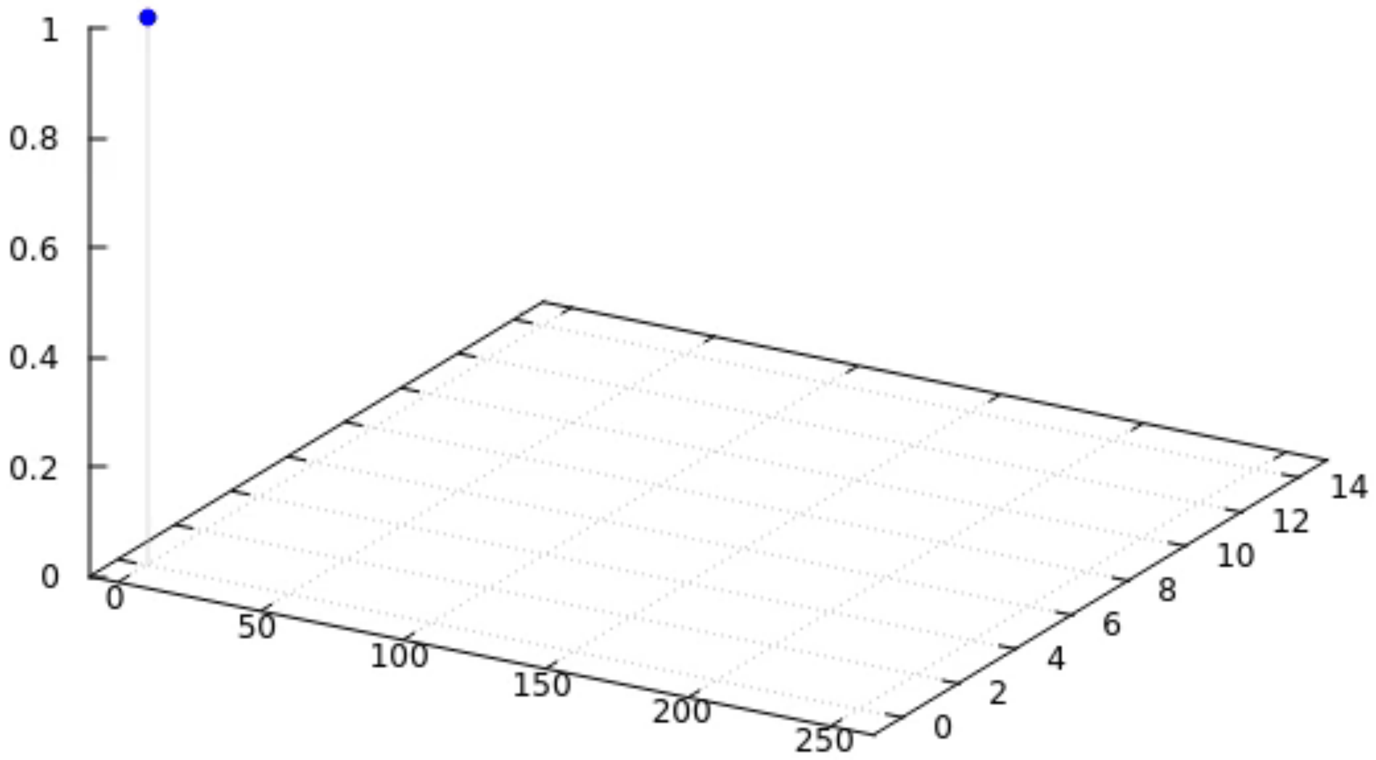
dove si ricordano le espressioni in binario

$$64 \rightarrow 01000000$$

$$128 \rightarrow 10000000$$

$$192 \rightarrow 11000000$$

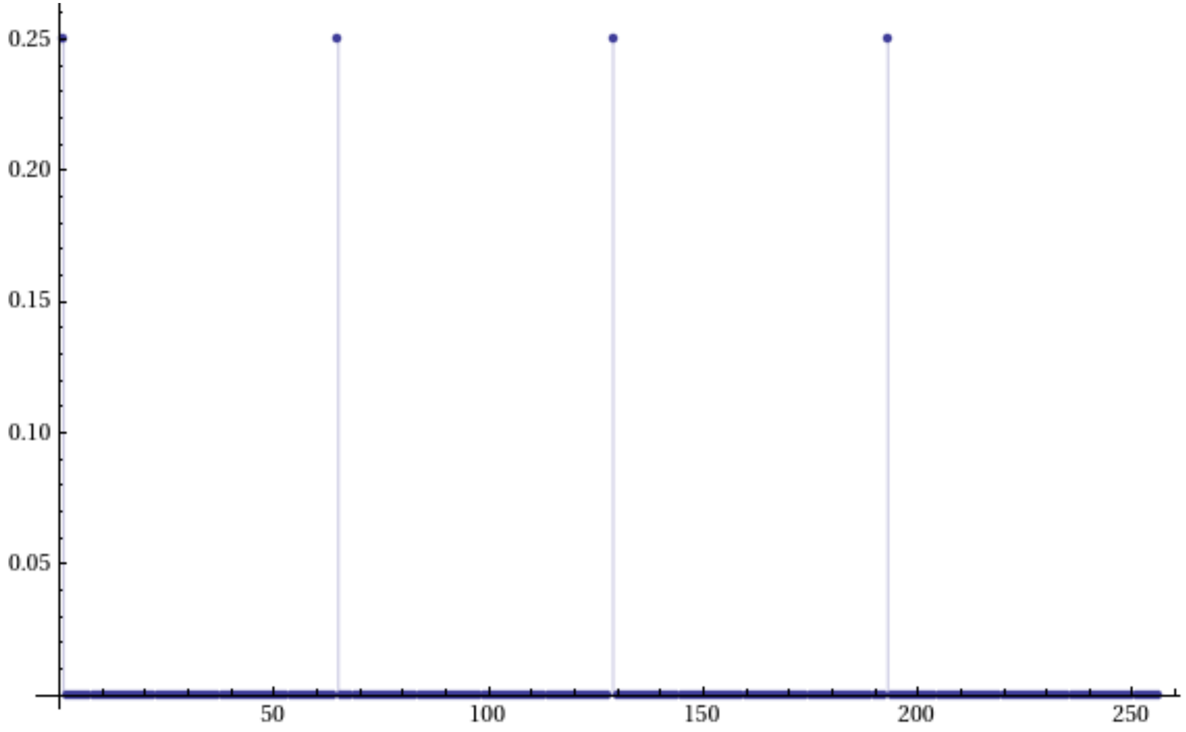
Factoring 15



Factoring 15

A questo punto l'algoritmo prevede la misura del primo registro: La distribuzione di probabilità marginale ottenuta è riassunta in figura:

$$P(c) = \sum_k P(c, k)$$



Quindi q = 64

e

Q/P = 256/64 = 4

Che è l'ordine cercato

Example: Factorize 799. Take m = 7.

We have to find the order P of the function f(a) = 7^a mod 799. (The answer is P = 368). We take n = 20

STEP 0: The initial state is

$$|\psi_0\rangle = |0\rangle|0\rangle. \tag{8.18}$$

STEP 1: The QFT on the first register results in

$$|\psi_1\rangle = \frac{1}{\sqrt{Q}} \sum_{x=0}^{Q-1} |x\rangle|0\rangle, \tag{8.19}$$

Example: Factorize 799. Take m = 7.

STEP 2: Application of U_f on $|\psi_1\rangle$ produces

$$|\psi_2\rangle = \frac{1}{\sqrt{Q}} \sum_{x=0}^{Q-1} |x\rangle |7^x \bmod 799\rangle$$
$$= \frac{1}{\sqrt{Q}} \Big[|0\rangle|1\rangle + |1\rangle|7\rangle + |2\rangle|49\rangle + |3\rangle|343\rangle + |4\rangle|4\rangle + |5\rangle|28\rangle$$
$$+ \ldots + |368\rangle|1\rangle + |369\rangle|7\rangle + |370\rangle|49\rangle + \ldots$$
$$+ |Q-2\rangle|756\rangle + |Q-1\rangle|498\rangle \Big]. \qquad (8.20)$$

Note that there are only $P = 368$ different states in the second register.

Example: Factorize 799. Take m = 7.

STEP 3: The QFT with $\omega = e^{2\pi i/Q}$, $Q = 2^n$, is applied to the first register. This results in

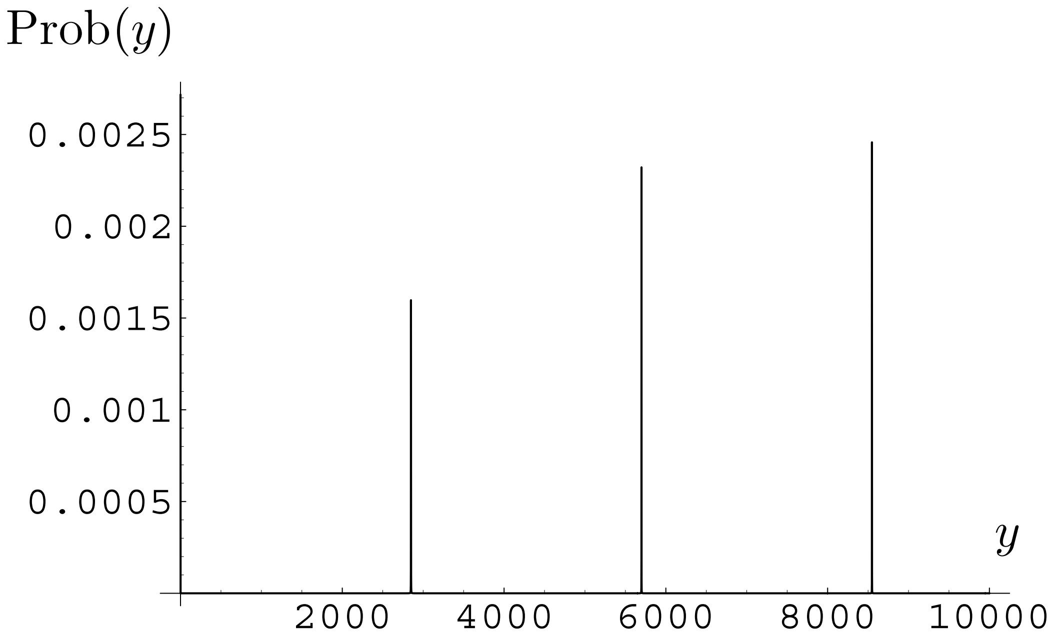
$$|\psi_3\rangle = \frac{1}{\sqrt{Q}} \sum_{x=0}^{Q-1} \frac{1}{\sqrt{Q}} \sum_{y=0}^{Q-1} \omega^{-xy} |y\rangle |7^x \bmod 799\rangle \equiv \frac{1}{Q} \sum_{y=0}^{Q-1} |y\rangle |\Upsilon(y)\rangle,$$

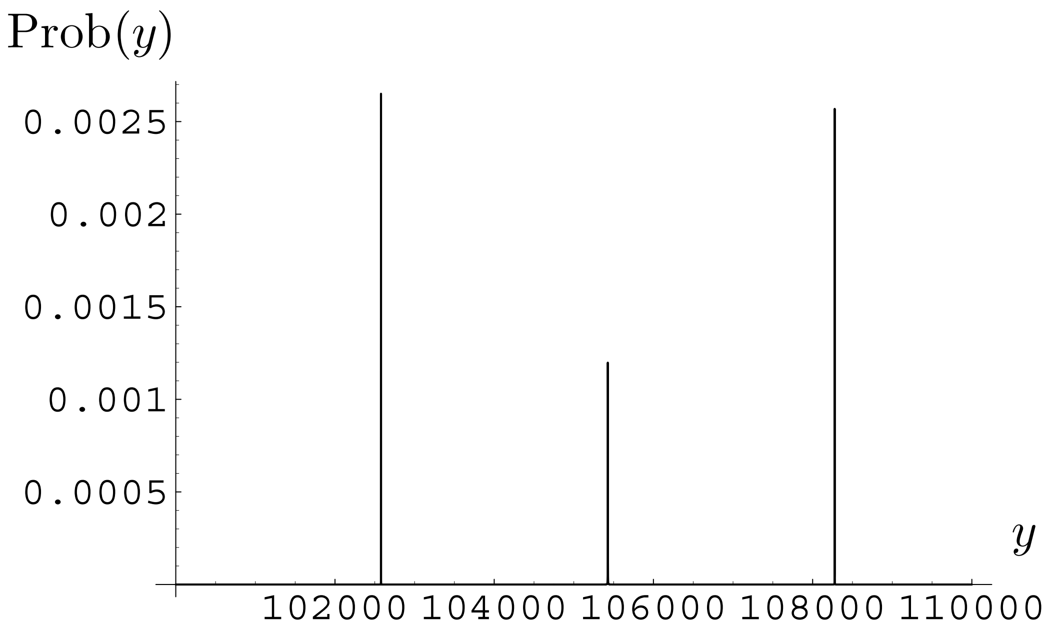
where

$$|\Upsilon(y)\rangle = \sum_{x=0}^{Q-1} \omega^{-xy} |7^x \bmod 799\rangle = \sum_{x=0}^{Q-1} e^{-2\pi i xy/Q} |7^x \bmod 799\rangle$$

$$
\begin{aligned}
|\Upsilon(y)\rangle &= \sum_{x=0}^{Q-1} e^{-2\pi i x y/Q} |7^x \bmod 799\rangle \\
&= |1\rangle + \omega^{-y}|7\rangle + \omega^{-2y}|49\rangle + \omega^{-3y}|343\rangle + \ldots \\
&\quad +\omega^{-368y}|1\rangle + \omega^{-369y}|7\rangle + \omega^{-370y}|49\rangle + \omega^{-371y}|343\rangle + \ldots \\
&\qquad\qquad + \ldots + \\
&\quad +\omega^{-736y}|1\rangle + \omega^{-737y}|7\rangle + \omega^{-738y}|49\rangle + \omega^{-739y}|343\rangle + \ldots \\
&\qquad\qquad + \ldots + \\
&\quad +\omega^{-1048432y}|1\rangle + \omega^{-1048433y}|7\rangle + \omega^{-1048434y}|49\rangle + \omega^{-1048435y}|343\rangle \\
&\qquad \ldots + \omega^{-1048575y}|498\rangle \\
&= (1 + \omega^{-368y} + \omega^{-736y} + \ldots + \omega^{-1048432y})|1\rangle \\
&\quad +(\omega^{-y} + \omega^{-369y} + \omega^{-737y} + \ldots + \omega^{-1048433y})|7\rangle \\
&\quad +(\omega^{-2y} + \omega^{-370y} + \omega^{-738y} + \ldots + \omega^{-1048434y})|49\rangle \\
&\quad +(\omega^{-3y} + \omega^{-371y} + \omega^{-739y} + \ldots + \omega^{-1048435y})|343\rangle \\
&\qquad\qquad + \ldots \\
&\quad +(\omega^{-87y} + \omega^{-455y} + \omega^{-823y} + \ldots)|794\rangle. \qquad (8.22)
\end{aligned}
$$

There are P = 368 ket vectors in the above expansion.





The coefficient of each vector becomes sizeable when and only when y is approximately a multiple of 2849. That means that q ∼ 2849 (in general r ≠ 0) and therefore P ∽ Q/2849 ∽ 368.0505. The order thus obtained is probabilistic, and its plausibility must be checked. This strategy is not practical when N is considerably large. There is a powerful method of continued fraction expansion by which we find the order P with a single measurement of the first register.