

Bell's theorem

Setting

Let us consider two $1/2$ -spin particles.



Two separated particles, upon which spin measurements can be performed.

a and **b** are the two directions of spin-measurement
A and B are the two outcomes, which can be either Y or N (up and down; + or - ...)

Any theory of nature must accommodate this setting, otherwise it is not a theory of nature, since such a situation occurs in nature.

Setting

$$p(A, B | \mathbf{a}, \mathbf{b}, \lambda)$$

This is the probability that in a measurement of spin of the left particle along direction \mathbf{a} the outcome is A, and in a measurement of spin of the right particle along direction \mathbf{b} the outcome is B. (p can also be 0 or 1, if the theory is deterministic)

λ is the state of the two-particle system.

Classical mechanics: λ = positions and momenta of the particles

Quantum mechanics: λ = wave function

Bohmian mechanics: λ = wave function and positions of the particles

We are not committing to any specific theory

Setting

We also define

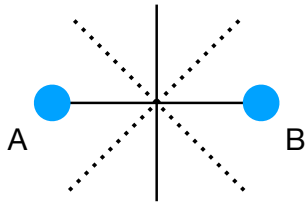
$p(A | \mathbf{a}, \lambda)$ = same as before, but with no measurement B

$p(B | \mathbf{b}, \lambda)$ = same as before, but with no measurement A

The definition of conditional probability implies:

$$p(A, B | \mathbf{a}, \mathbf{b}, \lambda) = p(A | B, \mathbf{a}, \mathbf{b}, \lambda) \cdot p(B | \mathbf{a}, \mathbf{b}, \lambda)$$

Bell's locality



$$p(A | B, \mathbf{a}, \mathbf{b}, \lambda) = p(A | \mathbf{a}, \lambda)$$

$$p(B | A, \mathbf{a}, \mathbf{b}, \lambda) = p(B | \mathbf{b}, \lambda)$$

When the two measurements are space-like separated from each other, what happens on one side cannot influence the other side.

Together with the rules of conditional probability, locality implies

$$p(A, B | \mathbf{a}, \mathbf{b}, \lambda) = p(A | \mathbf{a}, \lambda) \cdot p(B | \mathbf{b}, \lambda)$$

The theorem

Define:

$$E_\lambda(\mathbf{a}, \mathbf{b}) = p(Y, Y | \mathbf{a}, \mathbf{b}, \lambda) + p(N, N | \mathbf{a}, \mathbf{b}, \lambda) - p(Y, N | \mathbf{a}, \mathbf{b}, \lambda) - p(N, Y | \mathbf{a}, \mathbf{b}, \lambda)$$

Sum of agreements minus sun of disagreements.

Then, using Bell's locality condition:

$$E_\lambda(\mathbf{a}, \mathbf{b}) = [p(Y | \mathbf{a}, \lambda) - p(N | \mathbf{a}, \lambda)][p(Y | \mathbf{b}, \lambda) - p(N | \mathbf{b}, \lambda)]$$


And:

$$E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d}) = [p(Y | \mathbf{a}, \lambda) - p(N | \mathbf{a}, \lambda)][p(Y | \mathbf{b}, \lambda) - p(N | \mathbf{b}, \lambda)] - [p(Y | \mathbf{a}, \lambda) - p(N | \mathbf{a}, \lambda)][p(Y | \mathbf{d}, \lambda) - p(N | \mathbf{d}, \lambda)]$$

The theorem

$$E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d}) = \underbrace{[p(Y|\mathbf{a}, \lambda) - p(N|\mathbf{a}, \lambda)]}_{= 1 - 2p(N|\mathbf{a}, \lambda)} [(p(Y|\mathbf{b}, \lambda) - p(N|\mathbf{b}, \lambda)) - (p(Y|\mathbf{d}, \lambda) - p(N|\mathbf{d}, \lambda))]$$

Therefore:

$$|E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| \leq \underbrace{|(p(Y|\mathbf{b}, \lambda) - p(N|\mathbf{b}, \lambda)) - (p(Y|\mathbf{d}, \lambda) - p(N|\mathbf{d}, \lambda))|}_{\mathbf{r}} \underbrace{|(p(Y|\mathbf{b}, \lambda) - p(N|\mathbf{b}, \lambda)) + (p(Y|\mathbf{d}, \lambda) - p(N|\mathbf{d}, \lambda))|}_{\mathbf{s}}$$


$$|E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| \leq \underbrace{|(p(Y|\mathbf{b}, \lambda) - p(N|\mathbf{b}, \lambda)) + (p(Y|\mathbf{d}, \lambda) - p(N|\mathbf{d}, \lambda))|}_{\mathbf{r}} \underbrace{|(p(Y|\mathbf{b}, \lambda) - p(N|\mathbf{b}, \lambda)) - (p(Y|\mathbf{d}, \lambda) - p(N|\mathbf{d}, \lambda))|}_{\mathbf{s}}$$

The theorem

So:

$$|E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| + |E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| \leq |r - s| + |r + s|$$

Taking the square we have:

$$[|r - s| + |r + s|]^2 = 2r^2 + 2s^2 + 2|r^2 - s^2|$$

which is either equal to $4r^2$ or to $4s^2$; in either case, it is less than or equal to 4, since $r, s \in [-1, +1]$. So:

$$|r - s| + |r + s| \leq 2$$

The theorem

So we end up with:

$$|E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| + |E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| \leq 2$$

This is Bell's inequality, which is the direct consequence of Bell's locality condition alone

$$p(A, B | \mathbf{a}, \mathbf{b}, \lambda) = p(A | \mathbf{a}, \lambda) \cdot p(B | \mathbf{b}, \lambda)$$



$$|E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| + |E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| \leq 2$$

The testability

Problem: not always we have full control of the state of the system (λ).
Therefore the above inequalities are not always testable.

Solution:

$\lambda = (\mu, \nu)$, where μ are controllable and ν are uncontrollable degrees of freedom

$$E_\mu(\mathbf{a}, \mathbf{b}) = \int E_{(\mu, \nu)}(\mathbf{a}, \mathbf{b}) \rho(\nu) d\nu$$

This is a physically measurable quantity

↑
Probability distribution; it reflects our ignorance

$$|E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| + |E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| \leq 2$$

The testability

Then:

$$\begin{aligned} |E_\mu(\mathbf{a}, \mathbf{b}) - E_\mu(\mathbf{a}, \mathbf{d})| + |E_\mu(\mathbf{c}, \mathbf{b}) + E_\mu(\mathbf{c}, \mathbf{d})| &\leq \\ &\leq \int d\nu \rho(\nu) \left[|E_{(\mu,\nu)}(\mathbf{a}, \mathbf{b}) - E_{(\mu,\nu)}(\mathbf{a}, \mathbf{d})| + |E_{(\mu,\nu)}(\mathbf{c}, \mathbf{b}) + E_{(\mu,\nu)}(\mathbf{c}, \mathbf{d})| \right] \\ &\leq 2 \int d\nu \rho(\nu) = 2 \end{aligned}$$

The inequality still holds.

Application to QM

Let us consider a singlet state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

This state is rotationally invariant, so the spin relation above holds for any direction.

$$p_\lambda^{AB}(\mathbf{a}, \mathbf{b} | Y, Y) = p_\lambda^{AB}(\mathbf{a}, \mathbf{b} | N, N) = \frac{1}{2} \sin^2 \frac{\theta_{\mathbf{a},\mathbf{b}}}{2}$$

$$p_\lambda^{AB}(\mathbf{a}, \mathbf{b} | Y, N) = p_\lambda^{AB}(\mathbf{a}, \mathbf{b} | N, Y) = \frac{1}{2} \cos^2 \frac{\theta_{\mathbf{a},\mathbf{b}}}{2}$$

$$\text{Then: } E_\lambda^{AB}(\mathbf{a}, \mathbf{b}) = -\cos \theta_{\mathbf{a},\mathbf{b}}$$

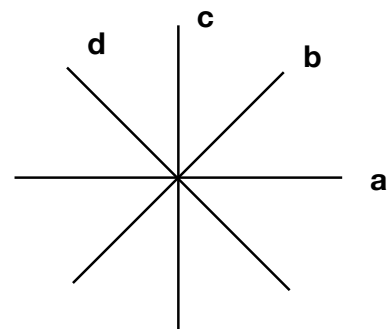
Application to QM

Then:

$$\begin{aligned}
 & |E_\lambda(\mathbf{a}, \mathbf{b}) - E_\lambda(\mathbf{a}, \mathbf{d})| + |E_\lambda(\mathbf{c}, \mathbf{b}) + E_\lambda(\mathbf{c}, \mathbf{d})| = \\
 & = |\cos \theta_{\mathbf{a}, \mathbf{b}} - \cos \theta_{\mathbf{a}, \mathbf{d}}| + |\cos \theta_{\mathbf{c}, \mathbf{b}} + \cos \theta_{\mathbf{c}, \mathbf{d}}|
 \end{aligned}$$

Let us choose the four angles as in the picture

$$\begin{aligned}
 & = \left| \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2} \right) \right| + \left| \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right| \\
 & = 2\sqrt{2} \geq 2
 \end{aligned}$$



The inequality is violated. **QM is nonlocal**

Nonlocality in QM

Where is the source of the nonlocality in QM?

Let us go back to the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]$$

A makes a measurement along the direction **a**. With probability 1/2 the outcome is Y and the state changes to

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle] \longrightarrow |\uparrow_{\mathbf{a}}\downarrow_{\mathbf{a}}\rangle$$

Also the state of the other particles has changed, no matter how far it is. This is the source on nonlocality.

Nonlocality in Physics

- Freedman and Clauser (1972)
- **Aspect et al. (1982)**
- Tittel et al. (1998)
- Weihs et al. (1998): experiment under "strict Einstein locality" conditions
- Pan et al. (2000) experiment on the GHZ state
- Rowe et al. (2001): the first to close the detection loophole
- Gröblacher et al. (2007) test of Leggett-type non-local realist theories
- Salart et al. (2008): separation in a Bell Test
- Ansmann et al. (2009): overcoming the detection loophole in solid state
- Giustina et al. (2013), Larsson et al (2014): overcoming the detection loophole for photons
- Christensen et al. (2013): overcoming the detection loophole for photons
- Hensen et al., Giustina et al., Shalm et al. (2015): "loophole-free" Bell tests
- Schmied et al. (2016): Detection of Bell correlations in a many-body system
- Handsteiner et al. (2017): "Cosmic Bell Test" - Measurement Settings from Milky Way Stars
- Rosenfeld et al. (2017): "Event-Ready" Bell test with entangled atoms and closed detection and locality loopholes
- The BIG Bell Test Collaboration (2018): "Challenging local realism with human choices"
- Rauch et al (2018): measurement settings from distant quasars