13 dic

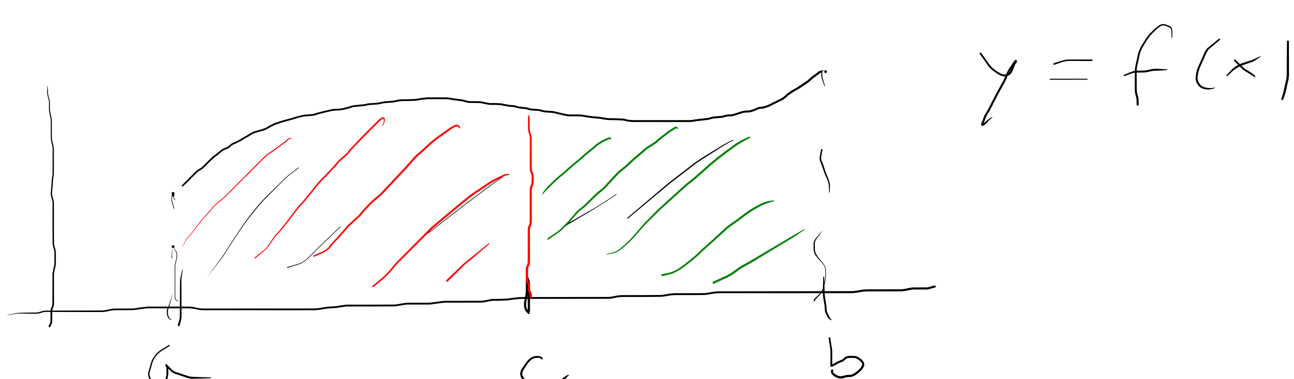
Teor Data $f:[a,b]\to\mathbb{R}$, data $c\in(a,b)$, sono equivalenti:

1) $f\in L[a,b]$

2) $f|_{[a,c]}\in L[a,c]$ e $f|_{[c,b]}\in L[c,b]$

Ed inoltre, quando 1) e 2) ~~sono~~ vere si ha

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$$



Def Sia I un intervallo, $f : I \to \mathbb{R}$ si dice localmente integrabile in I se per ogni coppia $a < b$ in I, $f|_{[a,b]} \in L[a,b]$. ~~Esprimiamo~~

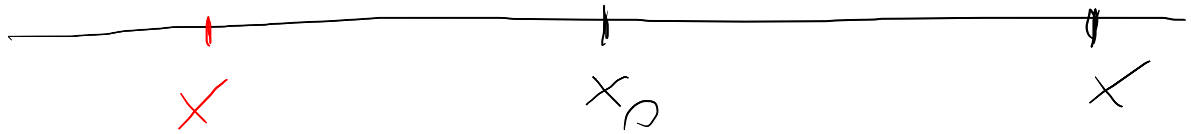
Esprimiamo la locale integrabilità scrivendo $f \in L_{loc}(\mathbb{R})$

Es $f \in C^0(\mathbb{R})$ si ha che $f \in L[a,b]$, cioè è integrabile in qualsiasi intervallo chiuso e limitato $[a,b] \subseteq \mathbb{R}$.

$$f \in C^0(\mathbb{R}) \Longrightarrow f \in L_{loc}(\mathbb{R}).$$

Def Sia I un intervallo, sia $x_0 \in I$, e sia $f \in L_{loc}(I)$

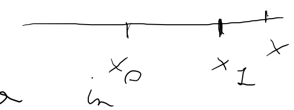
Per $x \in I$



$$F(x) = \int_{x_0}^{x} f(t)\,dt = \begin{cases} \text{è l'integrale di Darboux in } [x_0, x] \text{ se } x > x_0 \\ 0 \quad \text{se} \quad x = x_0 \\ -\int_{x}^{x_0} f(t)\,dt \quad \text{se} \quad x < x_0 \end{cases}$$

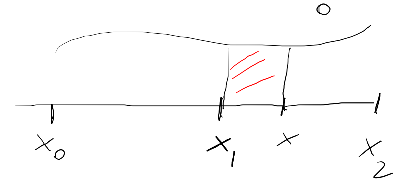
Lemma Se $f \in L_{loc}(I)$ allora $F \in C^0(I)$.

Dim Ricordiamo che $F(x) = \int_{x_0}^{x} f(t)\,dt$



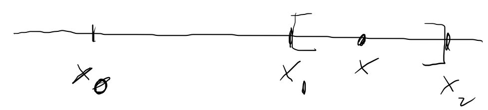
Verifichiamo che se $x_1 \in I$, F è continua in x_1. Devo cioè dimostrare che $\lim_{x \to x_1^+} \int_{x_0}^{x} f(t)\,dt = \int_{x_0}^{x_1} f(t)\,dt$

$$F(x) - F(x_1) = \int_{x_0}^{x} f(t)\,dt - \int_{x_0}^{x_1} f(t)\,dt$$



$$= \cancel{\int_{x_0}^{x_1} f(t)\,dt} + \int_{x_1}^{x} f(t)\,dt - \cancel{\int_{x_0}^{x_1} f(t)\,dt}$$

$$F(x) - F(x_1) = \int_{x_1}^{x} f(t)\,dt$$



Considero $x_2 \in I$ t.c. $x_2 > x > x_1$

Per ipotesi $f|_{[x_1,x_2]} \in L[x_1,x_2]$. Pertanto esiste una costante M t.c. $|f(t)| \le M \quad \forall t \in [x_1,x_2]$.

$$|F(x) - F(x_1)| = \left| \int_{x_1}^{x} f(t)\,dt \right| \quad \text{dove} \quad x_1 < x < x_2$$

$$\le \int_{x_1}^{x} |f(t)|\,dt \le \int_{x_1}^{x} M\,dt = M(x - x_1) \xrightarrow{x \to x_1} 0$$

$$0 \le |F(x) - F(x_1)| \le M(x - x_1) \xrightarrow{x \to x_1^+} 0$$

$$\Rightarrow \lim_{x \to x_1^+} F(x) = F(x_1)$$

Teor Sia $f \in L_{loc}(I)$, $x_0 \in I$, $F(x) = \int_{x_0}^{x} f(t)\,dt$

Sia $x_1 \in I$

Supponiamo che abbia senso e sia finito $\lim_{x \to x_1^+} f(x) = f(x_1^+)$

Allora $F'_d(x_1) = f(x_1^+)$

Se ha senso ed è finito $\lim_{x \to x_1^-} f(x) = f(x_1^-)$

allora $F'_s(x_1) = f(x_1^-)$

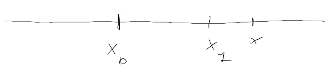
In particolare, se $f(x_1^+) = f(x_1^-)$ allora

$$F'(x_1) = f(x_1^+) = f(x_1^-)$$

Ancora più in particolare, se $f(x_1^+) = f(x_1^-) = f(x_1)$

allora $F'(x_1) = f(x_1)$.

Dim



Partiamo da $\lim_{x \to x_1^+} f(x) = f(x_1^+) \in \mathbb{R}$. Dimostreremo che

$$\lim_{x \to x_1^+} \left[\frac{F'(x) - F(x_1)}{x - x_1} - f(x_1^+) \right] = 0$$

$$\frac{F(x) - F(x_1)}{x - x_1} - f(x_1^+) = \frac{\int_{x_0}^{x} f(t)\,dt - \int_{x_0}^{x_1} f(t)\,dt}{x - x_1} - f(x_1^+)$$

$$= \frac{\cancel{\int_{x_0}^{x_1} f(t)\,dt} + \int_{x_1}^{x} f(t)\,dt - \cancel{\int_{x_0}^{x_1} f(t)\,dt}}{x - x_1} - f(x_1^+)$$

$$= \frac{1}{x - x_1} \int_{x_1}^{x} f(t)\,dt - f(x_1^+) \frac{x - x_1}{x - x_1} \qquad \left(\int_{x_1}^{x} f(x_1^+)\,dt = (x - x_1) f(x_1^+) \right)$$

$$= \frac{1}{x - x_1} \left[\int_{x_1}^{x} f(t)\,dt - \int_{x_1}^{x} f(x_1^+)\,dt \right] = \frac{1}{x - x_1} \int_{x_1}^{x} \left(f(t) - f(x_1^+) \right) dt$$

$$\left| \frac{F(x) - F(x_1)}{x - x_1} - f(x_1^+) \right| = \frac{1}{x - x_1} \left| \int_{x_1}^{x} \left(f(t) - f(x_1^+) \right) dt \right|$$

$$\leq \frac{1}{x - x_1} \int_{x_1}^{x} \left| f(t) - f(x_1^+) \right| dt$$

Siccome $\lim_{t \to x_1^+} f(t) = f(x_1^+)$ so che

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ t.c. $0 < t - x_1 < \delta_\varepsilon \Rightarrow |f(t) - f(x_1^+)| < \varepsilon$

Se considero il δ_ε della precedente riga ho che per $0 < x - x_1 < \delta_\varepsilon$

$$\left| \frac{F(x) - F(x_1)}{x - x_1} - f(x_1^+) \right| \leq \frac{1}{x - x_1} \int_{x_1}^{x} \underbrace{\left| f(t) - f(x_1^+) \right|}_{< \varepsilon} dt < \underbrace{\frac{1}{x - x_1} \int_{x_1}^{x} \varepsilon \, dt}_{= \varepsilon}$$

Abbiamo dimostrato che

$\forall \varepsilon > 0 \ \exists \ \delta_\varepsilon > 0$ t.c. $0 < x - x_1 < \delta_\varepsilon$

$$\Rightarrow \left| \frac{F(x) - F(x_1)}{x - x_1} - f(x_1^+) \right| < \varepsilon$$

$$\Rightarrow \lim_{x \to x_1^+} \left[\frac{F(x) - F(x_1)}{x - x_1} - f(x_1^+) \right] = 0$$

$$\Longleftrightarrow \quad F'(x_1) = f(x_1^+)$$

<u>Corollario</u> Se $f \in C^0(I)$ e $x_0 \in I$, allora posto

$F(x) = \int_{x_0}^{x} f(t)\,dt$ si ha $F'(x) = f(x)$ e quindi

$F \in C^1(I)$

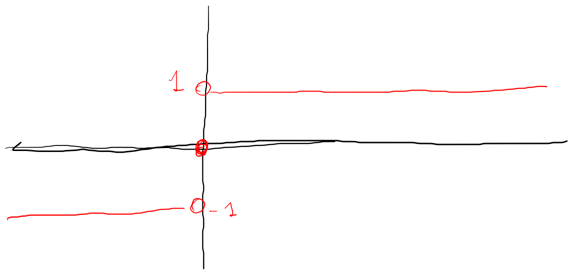
__Def__ Una funzione $f : I \longrightarrow \mathbb{R}$ è primitivabile in I se esiste una $G : I \longrightarrow \mathbb{R}$ t.c. $G'(x) = f(x) \quad \forall x \in I$.

G viene detto una primitiva di f.

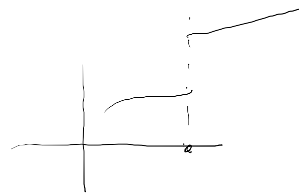
__Osservazione__ 1 Se G è una primitiva di f in (a,b) allora anche $G + c$ è una primitiva di f in (a,b) per qualsiasi $c \in \mathbb{R}$.

2) $f \in C^0((a,b))$ è primitivabile in (a,b). Segue dal fatto che se $x_0 \in (a,b)$ ~~e~~ se pongo $F(x) = \int_{x_0}^{x} f(t)\,dt$ per il teor fond del calcolo si ha $F'(x) = f(x) \quad \forall x \in (a,b)$.

Esempio $\operatorname{sign}(x) = \begin{cases} 1 & \text{se } x > 0 \\ 0 & \text{se } x = 0 \\ -1 & \text{se } x < 0 \end{cases}$



$\operatorname{sign}(x)$ non è primitivabile in $\mathbb{R}$.



Notiamo che $\operatorname{sign}(0^+) = \lim_{x \to 0^+} \operatorname{sign}(x) = \lim_{x \to 0^+} 1 = 1$

$$\operatorname{sign}(0^-) = \lim_{x \to 0^-} \operatorname{sign}(x) = \lim_{x \to 0^-} (-1) = -1$$

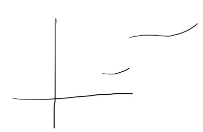
Supponiamo per assurdo che esista una primitiva $G(x)$ in $\mathbb{R}$.

Per ogni $x \in \mathbb{R}$ $\quad G'(x) = \operatorname{sign}(x)$

$$G'_d(0) = \lim_{x \to 0^+} \frac{G(x) - G(0)}{x} = \lim_{x \to 0^+} G'(x)$$

$$= \lim_{x \to 0^+} \operatorname{sign}(x) = \lim_{x \to 0^+} 1 = 1$$

$$G'_d(0) = 1$$

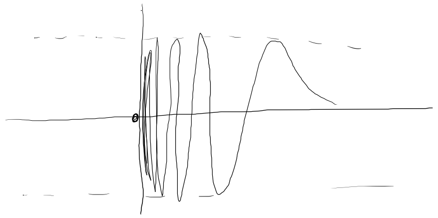


$$G'_s(0) = \lim_{x \to 0^-} \frac{G(x) - G(0)}{x} = \lim_{x \to 0^-} G'(x)$$

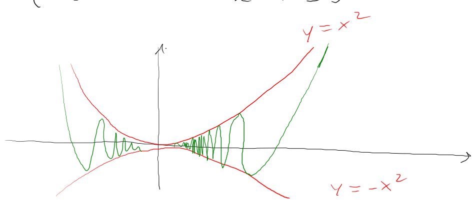
$$= \lim_{x \to 0^-} \operatorname{sign}(x) = \lim_{x \to 0^-} (-1) = -1$$

$G'_s(0) = -1 \implies G'(0)$ non esiste.

$$f(x)=\begin{cases}\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$$ è primitivabile



$$G(x)=\begin{cases}x^2\cos\left(\frac{1}{x}\right) & \text{se } x\neq 0\\ 0 & \text{se } x=0\end{cases}$$



$$G'(x)=\left(x^2\cos\left(\tfrac{1}{x}\right)\right)'=\left(x^2\right)'\cos\left(\tfrac{1}{x}\right)+x^2\left(\cos\left(\tfrac{1}{x}\right)\right)'$$

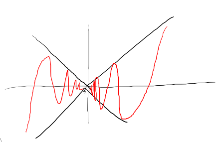
$$=2x\cos\left(\tfrac{1}{x}\right)+x^2\cos'\left(\tfrac{1}{x}\right)\left(\tfrac{1}{x}\right)'=$$

$$=2x\cos\left(\tfrac{1}{x}\right)+\cancel{x^2}(-1)\sin\left(\tfrac{1}{x}\right)\left(-\tfrac{1}{\cancel{x^2}}\right)$$

$$=2x\cos\left(\tfrac{1}{x}\right)+\sin\left(\tfrac{1}{x}\right)$$

$$G'(0)=\lim_{x\to 0}\frac{G(x)-G(0)}{x}=\lim_{x\to 0}\frac{G(x)}{x}=\lim_{x\to 0}\frac{x^{\cancel{2}}\cos\left(\frac{1}{x}\right)}{\cancel{x}}$$

$$=\lim_{x\to 0}x\cos\left(\tfrac{1}{x}\right)=0$$



$$\begin{cases}G'(x)=2x\cos\left(\tfrac{1}{x}\right)+\underbrace{\sin\left(\tfrac{1}{x}\right)}_{f}\\ G'(0)=0\end{cases}$$

$$k(x)=\begin{cases}2x\cos\left(\tfrac{1}{x}\right) & \text{se } x\neq 0\\ 0 & \text{se } x=0\end{cases}$$

$k\in C^0(\mathbb{R})$ ed ho

$G'(x)=k(x)+f(x)$ per ogni $x\in\mathbb{R}$.

$k\in C^0(\mathbb{R}) \Rightarrow k$ è primitivabile e quindi $\exists\ K$ in $\mathbb{R}$

t.c. $K'(x)=k(x)\quad\forall x\in\mathbb{R}$.

$$G'(x)=K'(x)+f(x)\Rightarrow f(x)=G'(x)-K'(x)$$

$G(x)-K(x)$ è una primitiva di $f(x)$. $\quad =\left(G(x)-K(x)\right)'$

Esercizio $F(x) = \int_0^x \operatorname{sen}\left(\frac{1}{t}\right) dt$ è una primitiva di $f(x)$.

Corollario (Teor. di Valutazione) Sia $f \in C^0([a,b])$ e sia $G \in C^1([a,b])$ una primitiva di f in $[a,b]$. Allora

$$\int_a^b f(x)\, dx = G(b) - G(a) = G(x)\Big]_a^b$$

Dim Sia $F(x) = \int_a^x f(t)\, dt$. Il teor. fond. del calcolo garantisce che $F'(x) = f(x) \quad \forall x \in [a,b]$

Ed ovviamente $F(a) = 0$

$$\int_a^b f(t)\, dt = F(b) = F(b) - F(a)$$

Consideriamo una altra primitiva $G(x)$ di $f(x)$

Allora esiste una costante c t.c. $G(x) = F(x) + c \quad \forall x \in [a,b]$.

$$G(b) - G(a) = (F(b) + \cancel{c}) - (F(a) + \cancel{c}) = F(b) - F(a)$$

$$= F(b) = \int_a^b f(t)\, dt$$

$$\left(\frac{x^3}{3}\right)' = x^2$$

$$\int_0^3 x^2\, dx = \frac{x^3}{3}\Big]_0^3 = \frac{3^3}{3} = 3^2 = 9$$

$f(x)$	$x^a \quad a \neq -1$	x^{-1}	e^x	$\cos x$	$\sin x$
primitive	$\frac{x^{a+1}}{a+1} + C$	$\lg\|x\| + C$	$e^x + C$	$\sin x + C$	$-\cos x + C$

$$\left(x^{a+1}\right)' = (a+1)\, x^a$$

$f(x)$	$\frac{1}{1+x^2}$	$\frac{1}{\sqrt{1-x^2}}$
primitive	$\arctan(x) + C$	$\arcsin x + C$

Calcolare i polinomi di McLaurin di arctan x

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$\sum_{j=0}^{n} x^j = \frac{1-x^{n+1}}{1-x}$$

$$\frac{1}{1-x} = \sum_{j=0}^{n} x^j + \frac{x^{n+1}}{1-x} = \sum_{j=0}^{n} x^j + o(x^n)$$

$$\lim_{x\to 0} \frac{\frac{x^{\cancel{n}+1}}{1-x}}{\cancel{x^n}} = \lim_{x\to 0} \frac{x}{1-x} = 0$$

$$\frac{1}{1+x} = \sum_{j=0}^{n} (-1)^j x^j + o(x^n) \quad , \quad \frac{1}{1+x^2} = \sum_{j=0}^{n} (-1)^j x^{2j} + o(x^{2n})$$

$$\operatorname{artg}(x) = \int_0^x \operatorname{artg}'(t)\, dt = \int_0^x \frac{1}{1+t^2}\, dt =$$

$$= \sum_{j=0}^{n} (-1)^j \int_0^x t^{2j} + \int_0^x o(t^{2n})\, dt$$

$$= \sum_{j=0}^{n} (-1)^j \left.\frac{t^{2j+1}}{2j+1}\right]_0^x + \int_0^x o(t^{2n})\, dt$$

$$= \sum_{j=0}^{n} (-1)^j \frac{x^{2j+1}}{2j+1} + \underbrace{\int_0^x o(t^{2n})\, dt}_{o(x^{2n+1})\ ?}$$

$2n+1$

Dimostriamo che quindi ho $\boxed{\int_0^x o(t^{2n})\, dt = o(x^{2n+1})}$

Si tratta di dimostrare che

$$\lim_{x\to 0} \frac{\int_0^x o(t^{2n})\, dt}{x^{2n+1}} = \lim_{x\to 0} \frac{o(x^{2n})}{(2n+1)x^{2n}} =$$

$$= \frac{1}{2n+1} \lim_{x\to 0} \frac{o(x^{2n})}{x^{2n}} = \frac{1}{2n+1}\, 0 = 0$$

$$f(x) = \int_0^{1+x} \frac{1}{1+t^2}\, dt \qquad \frac{1}{1+t^2} = \sum_{j=0}^{n} (-1)^j t^{2j} + o(t^{2n})$$

$$= \sum_{j=0}^{n} (-1)^j \int_0^{1+x} t^{2j}\, dt + \int_0^{1+x} o(t^{2n})\, dt$$

$$= \sum_{j=0}^{n} (-1)^j \left.\frac{t^{2j+1}}{2j+1}\right]_0^{1+x} + \int_0^{1+x} o(t^{2n})\, dt$$

$$= \sum_{j=0}^{n} (-1)^j \frac{(1+x)^{2j+1}}{2j+1} + \int_0^{1+x} o(t^{2n})\, dt$$

$$f(x) = \int_0^{1+x} \frac{1}{1+t^2}\, dt = \underbrace{\int_0^{1} \frac{1}{1+t^2}\, dt}_{\arctan 1 = \frac{\pi}{4}} + \int_1^{1+x} \frac{1}{1+t^2}\, dt$$