14 Dicembre

Notazione: Data $f:(a,b)\to\mathbb{R}$ primitivabile la sua generica primitiva sarà denotata con $\int f(x)\,dx$.

Es
$$\int x^2\,dx = \frac{x^3}{3} + C$$

Proposizione: Sia I un intervallo, $f,g \in C^1(I)$. Vale la seguente formula
$$\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx \qquad (1)$$

Dato $[a,b]$ e $f,g \in C^1([a,b])$, vale
$$\int_a^b f'(x)\,g(x)\,dx = f(x)\,g(x)\Big]_a^b - \int_a^b f(x)\,g'(x)\,dx \qquad (2)$$

Dim: Ricordiamoci che $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

$$f'(x)\,g(x) = (fg)'(x) - f(x)\,g'(x)$$
$$\int f'(x)\,g(x)\,dx = \int \left[(fg)'(x) - f(x)\,g'(x)\right] dx$$
$$= \int (fg)'(x)\,dx - \int f(x)\,g'(x)\,dx$$
$$= f(x)\,g(x) - \int f(x)\,g'(x)\,dx$$

Conclusione: $\boxed{\int f'(x)\,g(x)\,dx = f(x)\,g(x) - \int f(x)\,g'(x)\,dx}$ (1)

Ora passiamo alla (2) che, ricordo, è
$$\int_a^b f'(x)\,g(x)\,dx = f(x)\,g(x)\Big]_a^b - \int_a^b f(x)\,g'(x)\,dx \qquad (2)$$

$$\int_a^b f'(x)\,g(x)\,dx \overset{\text{teo. di valutazione}}{=} \int f'(x)\,g(x)\,dx\Big]_a^b =$$
$$\overset{(1)}{=} \left(f(x)\,g(x) - \int f(x)\,g'(x)\,dx\right)\Big]_a^b =$$
$$= f(x)\,g(x)\Big]_a^b - \int f(x)\,g'(x)\,dx\Big]_a^b$$
$$= f(x)\,g(x)\Big]_a^b - \int_a^b f(x)\,g'(x)\,dx$$

$$\int_a^b f'(x)\,g(x)\,dx = f(x)\,g(x)\Big]_a^b - \int_a^b f(x)\,g'(x)\,dx \qquad (2)$$

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

$$\int x e^x \, dx = \int x \left(e^x\right)' dx = x e^x - \int (x)' e^x \, dx$$

$$= x e^x - \int e^x \, dx = x e^x - e^x + C.$$

$$\int P(x)\, e^x \, dx = \int P(x) \left(e^x\right)' dx = P(x)\, e^x - \int P'(x)\, e^x \, dx$$

$\sin x$

$\cos x$

$$\int e^x \operatorname{sen} x \, dx$$

$$\int e^{ax} \, dx = \frac{e^{ax}}{a} + C$$

$$(e^{ax})' = e^{ax} (ax)' = a e^{ax}$$

$$\left(\frac{e^{ax}}{a}\right)' = e^{ax}$$

$$\int e^x \sin x \, dx = \int (e^x)' \operatorname{sen} x \, dx = e^x \sin x - \int e^x (\operatorname{sen} x)' dx$$

$$= e^x \operatorname{sen}(x) - \int e^x \cos x \, dx$$

$$= e^x \operatorname{sen}(x) - \int (e^x)' \cos x \, dx$$

$$= e^x \sin x - \left[e^x \cos x - \int e^x (\cos x)' \, dx \right]$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \operatorname{sen}(x) \, dx \right]$$

$$\int e^x \sin(x) \, dx = e^x \operatorname{sen}(x) - e^x \cos(x) - \int e^x \sin(x) \, dx$$

$$\int e^x \operatorname{sen}(x) \, dx = \frac{e^x \sin(x) - e^x \cos(x)}{2} + C$$

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

$$\int \lg x \, dx = \int 1 \lg x \, dx = \int (x)' \lg x \, dx$$

$$= x \lg x - \int x (\lg x)' \, dx$$

$$= x \lg x - \int \cancel{x} \frac{1}{\cancel{x}} \, dx$$

$$\int \lg x \, dx = x \lg x - \int 1 \, dx = x \lg x - x + C$$

$$\int f'(x)\, g(x)\, dx = f(x)\, g(x) - \int f(x)\, g'(x)\, dx$$

$$\int \arctan(x)\, dx = \int 1 \arctan(x)\, dx =$$

$$= \int (x)' \arctan x = x \arctan x - \int x \,(\arctan x)'\, dx$$

$$= x \arctan x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx$$

$$\boxed{\int \arctan x\, dx = x \arctan x - \frac{1}{2} \lg(1+x^2) + C}$$

dove $\left(\lg(1+x^2)\right)' = \lg'(1+x^2) \quad (1+x^2)'$

$$= \frac{1}{1+x^2} \quad 2x$$

Lemma Siano I e J due ~~intervall~~ intervalli,

$u(x): I \longrightarrow J \qquad f(u): J \longrightarrow \mathbb{R}, \quad u \in C^1(I), \quad f(u) \in C^0(J).$

Vale l'uguaglianza

$$\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\, du\right)(u(x)) \qquad (1)$$

Osservazione Di solito (1) viene scritto nella form

$$\int f(u(x))\, u'(x)\, dx = \int f(u)\, du$$

Dim Per dimostrare $\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\, du\right)(u(x))$ (1)

$$\frac{d}{dx} \int f(u(x))\, u'(x)\, dx = f(u(x))\, u'(x)$$

$$\frac{d}{dx} \left(\int f(u)\, du\right)(u(x)) = \frac{d}{du} \int f(u)\, du \Big|_{u=u(x)} u'(x)$$

$$= f(u)\Big|_{u=u(x)} u'(x) = f(u(x))\, u'(x)$$

$$\int f(u(x))\,u'(x)\,dx = \int f(u)\,du$$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx =$$

$u = 1 + x^2$

$\frac{du}{dx} = 2x \qquad du = 2x\,dx$

$$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\lg|u| + C = \frac{1}{2}\lg(1+x^2) + C$$

$$= \lg\sqrt{1+x^2} + C$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

$u = \cos x \qquad du = -\sin x\,dx$

$\frac{du}{dx} = -\sin x$

$$= -\int \frac{du}{u} = -\lg|u| + C = -\lg|\cos x| + C$$

Lemma $[a,b]$, J; $u(x): [a,b] \to J$, $f: J \to \mathbb{R}$, $u \in C^1([a,b])$ e $f \in C^0(J)$. Allora vale

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

Dim Si parte da $\int f(u(x))\, u'(x)\, dx = \left(\int f(u)\, du\right)(u(x))$

$$\int_a^b f(u(x))\, u'(x)\, dx \overset{\text{Teor volutiy.}}{=} \int f(u(x))\, u'(x)\, dx \Big]_a^b =$$

$$= \left(\int f(u)\, du\right)(u(x)) \Big]_a^b =$$

$$= \left(\int f(u)\, du\right)(u(b)) - \left(\int f(u)\, du\right)(u(a))$$

$$= \int f(u)\, du \Big]_{u(a)}^{u(b)} \overset{\text{teor volutiy}}{=} \int_{u(a)}^{u(b)} f(u)\, du$$

$$\int_a^b f(u(x))\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,dx$$

$$\int_1^2 \frac{x}{1+x^2}\,dx =$$

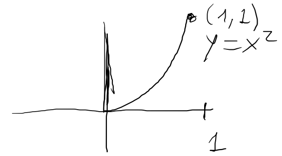
$u = 1 + x^2$

$du = 2x\,dx$

$$= \int_2^5 \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int_2^5 \frac{1}{u}\,du = \frac{1}{2}\lg u \Big]_2^5$$

$$= \frac{1}{2}\left(\lg 5 - \lg 2\right) = \frac{1}{2}\lg\frac{5}{2}$$

$$\int_0^1 \sqrt{1+4x^2}\,dx$$



$$u = 2x$$

$$du = 2\,dx$$

$$\frac{1}{2}\int_0^2 \sqrt{1+u^2}\,du = \frac{1}{2}\int_0^2 \sqrt{1+x^2}\,dx$$

$$ch^2 t - sh^2 t = 1 \qquad ch^2 t = 1 + sh^2 t$$

$$ch\,t = \sqrt{1+sh^2 t} \qquad x = sh\,t \qquad t = lg\left(x+\sqrt{x^2+1}\right)$$

$$dx = ch\,t\,dt$$

$$= \frac{1}{2}\int_0^{t_0} \sqrt{1+sh^2(t)}\;ch\,t\,dt \qquad t_0 = lg\left(2+\sqrt{5}\right)$$

$$= \frac{1}{2}\int_0^{t_0} ch^2 t\,dt = \frac{1}{2}\int_0^{t_0} \frac{ch(2t)+1}{2}\,dt =$$

Vale $ch^2 t = \dfrac{ch(2t)+1}{2}$

$$\begin{cases} \cos^2 x = \dfrac{1+\cos(2x)}{2} \\ \sin^2 x = \dfrac{1-\cos(2x)}{2} \end{cases}$$

$$= \frac{1}{4}\int_0^{t_0} ch(2t)\,dt + \frac{1}{4}\underbrace{\int_0^{t_0} dt}_{t_0}$$

$$= \frac{1}{8}\,sh(2t)\Big]_0^{t_0} + \frac{1}{4}\,lg\left(2+\sqrt{5}\right)$$

$$= \frac{1}{8}\,sh(2t_0) + \frac{1}{4}\,lg\left(2+\sqrt{5}\right) = \frac{1}{2}\sqrt{5} + \frac{1}{4}\,lg\left(2+\sqrt{5}\right)$$

$$sh(2t_0) = 2\,\underbrace{sh(t_0)}_{2}\,ch(t_0) = 4\sqrt{1+sh^2(t_0)} = 4\sqrt{5}$$

$$x = sh\,t$$

$$2 = sh\,t_0$$

Espansioni di Hermite. Esistono costanti A, B, C t.c.

$$\frac{1}{x^3-3x^2+2x} = \frac{1}{x\left(x^2-3x+2\right)} = \frac{1}{x\,(x-1)\,(x-2)} =$$

$$\frac{1}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$\int \frac{1}{x^3-3x^2+2x}\,dx = A\int \frac{1}{x}\,dx + B\int \frac{dx}{x-1} + C\int \frac{dx}{x-2} =$$

$$= A \lg|x| + B \lg|x-1| + C \lg|x-2| + k$$

$$R(x) = \frac{1}{x^3-3x^2+x} = \boxed{\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}} \quad *$$

Moltiplico per x

$$\frac{1}{(x-1)(x-2)} = A + x\left(\frac{B}{x-1} + \frac{C}{x-2}\right)$$

Per $x = 0$ ottengo $A = R(x)\, x \Big|_{x=0} = \frac{1}{(x-1)(x-2)}\Big|_{x=0}$

$$\boxed{A = \frac{1}{2}}$$

Moltiplico la formula* di per $(x-1)$

$$R(x)(x-1) = \frac{1}{x(x-2)} = B + (x-1)\left[\frac{A}{x} + \frac{C}{x-2}\right]$$

$$B = R(x)(x-1)\Big|_{x=1} = \frac{1}{x(x-2)}\Big|_{x=1} = -1$$

$$C = R(x)(x-2)\Big|_{x=2} = \frac{1}{x(x-1)}\Big|_{x=2} = \frac{1}{2}$$

$$\frac{1}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

Qui deve essere $A + B + C = 0$

$$\frac{x}{x^3-3x^2+2x} = A + B\frac{x}{x-1} + C\frac{x}{x-2} \xrightarrow[x\to+\infty]{}$$

$$0 = A + B + C$$

$$R(x) = \frac{x^2}{x^3-3x^2+2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$1 = A + B + C$$

$$A = R(x)\, x\Big|_{x=0} = \frac{x^2}{(x-1)(x-2)}\Big|_{x=0} = 0$$

$$B = \frac{x^2}{x(x-2)}\Big|_{x=1} = -1$$

$$C = \frac{x^2}{x(x-1)}\Big|_{x=2} = \frac{4}{2} = 2$$

$$0 - 1 + 2 = 1$$

$$\frac{x^3}{x^3 - 3x^2 + x} \neq \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$\downarrow x \to +\infty$$

$$1 \qquad = \qquad 0$$