16 Dic

$$\frac{x^3+x+1}{x(x-1)(x-2)} \neq \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$\frac{x^3+x+1}{x(x-1)(x-2)} \xrightarrow{x \to +\infty} 1 \qquad \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2} \xrightarrow{x \to +\infty} 0$$

razionale

$$R(z) = \frac{P(z)}{Q(z)}$$

$P(z)$, $Q(z)$ due polinomi

Consideremo a breve il caso grado $P <$ grado Q

Quando grado $P \geq$ grado Q applicheremo la divisione tra polinomi, cioè esistono polinomi $q(z)$ e $r(z)$

$$P(z) = Q(z)\, q(z) + r(z)$$

$q(z)$ quoziente, $r(z)$ resto

grado $r <$ grado Q

$$\frac{P(z)}{Q(z)} = \frac{Q(z)\, q(z) + r(z)}{Q(z)} = q(z) + \frac{r(z)}{Q(z)}$$

Teorema Sia $R(z) = \frac{P(z)}{Q(z)}$ grado $P <$ grado Q

$Q(z) = q\,(z-z_1)^{m_1} \cdots (z-z_k)^{m_k}$ dove $z_1, \dots, z_k \in \mathbb{C}$ sono distinti tra loro, $m_1 + \dots + m_k =$ grado Q

$m_1, \dots, m_k \in \mathbb{N}$. Allora esistono delle costanti in $\mathbb{C}$

$$\begin{array}{ll} z_1 & A_{11}, \dots, A_{1m_1} \\ \vdots & \\ z_k & A_{k1}, \dots, A_{km_k} \end{array}$$

tali che

$$R(z) = \frac{A_{11}}{z-z_1} + \dots + \frac{A_{1m_1}}{(z-z_1)^{m_1}} + \dots + \frac{A_{k1}}{z-z_k} + \dots + \frac{A_{km_k}}{(z-z_k)^{m_k}}$$

$$A_{j\ell} = \frac{1}{(m_j-\ell)!} \left(\frac{d}{dz}\right)^{m_j-\ell} R(z)(z-z_j)^{m_j} \Big|_{z=z_j}$$

$$\frac{A_{j\ell}}{(z-z_j)^\ell}$$

Esempio $R(x) = \frac{1}{x^3-3x^2+2x} = \frac{\frac{1}{2}}{x} - \frac{1}{x-1} + \frac{\frac{1}{2}}{x-2}$

$$\frac{1}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}$$

$$A = R(x)\,x \Big|_{x=0} = \frac{1}{(x-1)(x-2)}\Big|_{x=0} = \frac{1}{2}$$

$$B = R(x)(x-1)\Big|_{x=1} = \frac{1}{x(x-2)}\Big|_{x=1} = -1$$

$$C = \frac{1}{2} = R(x)(x-2)\Big|_{x=2} = \frac{1}{x(x-1)}\Big|_{x=2} = \frac{1}{2}$$

$$R(z) = \frac{1}{(z^2+1)^2}$$

$$(z^2+1)^2 = \left((z-i)(z+i)\right)^2 = (z-i)^2 (z+i)^2$$

$$z^2+1 = (z-i)(z+i)$$

$$R(z) = \frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{(z-i)^2} + \frac{D}{(z+i)^2}$$

$$\frac{A_{j\ell}}{(z-z_j)^{\ell}} \qquad A_{j\ell} = \frac{1}{(m_j-\ell)!}\left(\frac{d}{dz}\right)^{m_j-\ell} R(z)(z-z_j)^{m_j}\Big|_{z=z_j}$$

$$C = R(z)(z-i)^2\Big|_{z=i} = \frac{1}{\cancel{(z-i)^2}(z+i)^2}\cancel{(z-i)^2}\Big|_{z=i}$$

$$= \frac{1}{(2i)^2} = -\frac{1}{4}$$

$$D = R(z)(z+i)^2\Big|_{z=-i} = \frac{1}{(z-i)^2}\Big|_{z=-i} = \frac{1}{(-2i)^2} = -\frac{1}{4}$$

$$A = \frac{1}{(2-1)!}\left(\frac{d}{dz}\right)^{2-1} R(z)(z-i)^2\Big|_{z=i} = \frac{d}{dz}\frac{1}{(z+i)^2}\Big|_{z=i} = -\frac{2}{(z+i)^3}\Big|_{z=i}$$

$$= \frac{-2}{(2i)^3} = -\frac{1}{4}\frac{1}{i^3} = \cancel{-}\frac{1}{4}\frac{1}{\cancel{-}i} = \frac{1}{4i} = \frac{-i}{4}$$

$$B = \frac{1}{(2-1)!}\left(\frac{d}{dz}\right)^{2-1} R(z)(z+i)^2\Big|_{z=-i} = \frac{d}{dz}\frac{1}{(z-i)^2}\Big|_{z=-i} = \frac{-2}{(z-i)^3}\Big|_{z=-i}$$

$$= \frac{-2}{(-2i)^3} = \frac{i}{4}$$

$$R(z) = \frac{1}{(z^2+1)^2} = \frac{-\frac{i}{4}}{z-i} + \frac{\frac{i}{4}}{z+i} - \frac{1}{4}\left(\frac{1}{(z-i)^2} + \frac{1}{(z+i)^2}\right)$$

$z = x$

$$\frac{1}{(x^2+1)^2} = \frac{-\frac{i}{4}}{x-i} + \frac{\frac{i}{4}}{x+i} - \frac{1}{4}\left(\frac{1}{(x-i)^2} + \frac{1}{(x+i)^2}\right)$$

$$= \frac{-i(x+i) + i(x-i)}{4\underbrace{(x-i)(x+i)}_{x^2+1}} - \frac{1}{4}\,\frac{(x+i)^2 + (x-i)^2}{(x^2+1)^2}$$

$$= \frac{\cancel{-ix} + 1 + \cancel{ix} + 1}{4(x^2+1)} - \frac{x^2 \cancel{+2ix} - 1 + x^2 \cancel{-2ix} - 1}{4(x^2+1)^2}$$

$$= \frac{2}{4(x^2+1)} - \frac{2}{4}\frac{(x^2-1)}{(x^2+1)^2} = \frac{1}{2(x^2+1)} - \frac{1}{2}\frac{x^2-1}{(x^2+1)^2}$$

$$\frac{1}{(x^2+1)^2} = \frac{1}{2(x^2+1)} - \frac{1}{4}\left[\frac{1}{(x-i)^2} + \frac{1}{(x+i)^2}\right]$$

$$= \frac{1}{2(x^2+1)} - \frac{1}{4}\left[\frac{d}{dx}\frac{(-1)}{x-i} + \frac{d}{dx}\frac{(-1)}{x+i}\right]$$

$$= \frac{1}{2(x^2+1)} + \frac{1}{4}\frac{d}{dx}\left[\frac{1}{x-i} + \frac{1}{x+i}\right] = \frac{1}{2(x^2+1)} + \frac{1}{4}\frac{d}{dx}\frac{x\cancel{+i}+x\cancel{-i}}{x^2+1} = \frac{1}{2(x^2+1)} + \frac{1}{2}\frac{d}{dx}\frac{x}{x^2+1}$$

$$R(x) = \frac{1}{(x^2+1)^2} = \frac{A}{x^2+1} + \frac{d}{dx} S(x) \qquad (1)$$

$\exists\ A \in \mathbb{R}$ e una funzione razionale $S(x)$

t.c. vale (1)

$$R(x) = \frac{P(x)}{Q(x)} \qquad \text{grado } Q > \text{grado } P$$

$$Q(x) = a\,(\cancel{x}-z_1)^{m_1}(\cancel{x}-z_2)^{m_2}\cdots(\cancel{x}-z_k)^{m_k}$$

$$= a\,(x-x_1)^{m_1}\cdots(x-x_L)^{m_L}(x-z_1)^{n_1}(x-\bar z_1)^{n_1}\cdots$$
$$\cdot(x-z_M)^{n_M}(x-\bar z_M)^{n_M}$$

$$= a\,(x-x_1)^{m_1}\cdots(x-x_L)^{m_L}\big((x-z_1)(x-\bar z_1)\big)^{n_1}$$
$$\cdots\big((x-z_M)(x-\bar z_M)\big)^{n_M}$$

$$(x-z_1)(x-\bar z_1) = x^2 + |z_1|^2 - x\bar z_1 - x z_1$$
$$= x^2 + |z_1|^2 - x\, 2\,\mathrm{Re}\, z_1$$

$z = x+iy \qquad z+\bar z = x+iy+x-iy = 2x$

Ogni polinomio $Q(x)$ a coefficienti reali può essere fattorizzato nella forma

$$Q(x) = a\,(x-x_1)^{m_1}\cdots(x-x_k)^{m_k}(x^2+b_1x+c_1)^{n_1}\cdots$$
$$(x^2+b_Lx+c_L)^{n_L}$$

dove $b_1,\dots,b_L,\ c_1,\dots,c_L$ sono dei numeri reali.

$$R(x) = \frac{P(x)}{Q(x)} \qquad \text{grado } P < \text{grado } Q$$

esiste una espressione della seguente forma

$$R(x) = \frac{A_1}{x-x_1} + \dots + \frac{A_k}{x-x_k} + \frac{B_1x+C_1}{x^2+b_1x+c_1} + \dots + \frac{B_Lx+C_L}{x^2+b_Lx+c_L} +$$

$$+ \frac{d}{dx} S(x)$$

per opportune costanti $A_1,\dots,A_k,\ B_1,\dots,B_L,\ C_1,\dots,C_L$
ed una opportuna funzione razionale $S(x)$