16 Dec $\quad 1 < p, q < \infty$

$$Tf(x) = \int_{\mathbb{R}^d} |x-y|^{-\gamma} f(y)\, dy$$

$$\frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d} \qquad 0 < \gamma < d$$

$$|Tf|_{L^q(\mathbb{R}^d)} \le C\, |f|_{L^p(\mathbb{R}^d)}$$

E. Lieb optimal constants

$$T S_{p\lambda} f = S_{q\lambda} T f \qquad f \ne 0$$

$(**)$

$\lambda_n \to +\infty$ $\qquad S_{p\lambda_n} f \to 0$ in $L^p(\mathbb{R}^d)$

$\lambda_n \to 0^+$

If T was compact $\exists\ g \in L^q(\mathbb{R}^d)$ s.t.

$$T S_{p\lambda_{n_k}} f \to g \quad \text{in } L^q(\mathbb{R}^d)$$

$g = 0$ by $(**)$

$$|T S_{p\lambda_{n_k}} f|_{L^q} \xrightarrow{n\to\infty} 0$$

$$\shortparallel$$

$$|S_{q\lambda_{n_k}} T f|_{L^q} = |Tf|_{L^q} > 0$$

Ex $\quad k \in L^1(\mathbb{T}^d) \Rightarrow$

$$Tf = k * f \qquad L^2(\mathbb{T}^d) \circlearrowleft$$

~~It~~ is compact

$$L^2(\mathbb{T}^d) \longrightarrow \ell^2(\mathbb{Z}^d)$$

$$(f : \mathbb{T}^d \to \mathbb{C}) \longrightarrow (\hat{f} : \mathbb{Z}^d \to \mathbb{C})$$

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-inx} \int_{\mathbb{T}^d} K(x-y) f(y)\, dy =$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-in(x-y)} e^{-iny} \int_{\cancel{\mathbb{R}}^d} K(x-y)\, \cancel{d} f(y)\, dy$$

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} dy\; e^{-iny} f(y) \int_{\mathbb{T}^d} e^{-in(x-\cancel{y})} K(x-\cancel{y})\, d\cancel{x}$$

$$\rightsquigarrow (2\pi)^d \hat{f}(n) \; \hat{K}(n)$$

$n = (n_1, \dots, n_d)$

$$T : L^2(\mathbb{T}^d) \circlearrowleft$$

$\partial_1 \rightsquigarrow -i n_1$

$$\ell^2(\mathbb{Z}^d) \circlearrowleft$$

$\hat{f} \longrightarrow \hat{f}\, \hat{K}$ is compact

$$K \in L^1(\mathbb{T}^d) \implies \hat{K} \in c_0(\mathbb{Z}^d) \subset \ell^\infty(\mathbb{Z}^d)$$

$$\lim_{n \to \infty} \hat{K}(n) = 0$$

$[-N, N]^d$ $(2N+1)^d$

$\ell^2(\mathbb{Z}^d)$

$$\hat{f}(n)\hat{K}(n) = \chi_{[-N,N]^d}(n)\; \hat{K}(n)\; \hat{f}(n) \; +$$

$$+ \left(1-\chi_{[-N,N]^d}(n)\right) \hat{k}(n) \hat{f}(n)$$

$\forall \varepsilon > 0 \quad \exists \quad N_\varepsilon \in \mathbb{N}$ s.t. $|n| > N_\varepsilon$

$$\Rightarrow \quad |\hat{k}(n)| < \varepsilon .$$

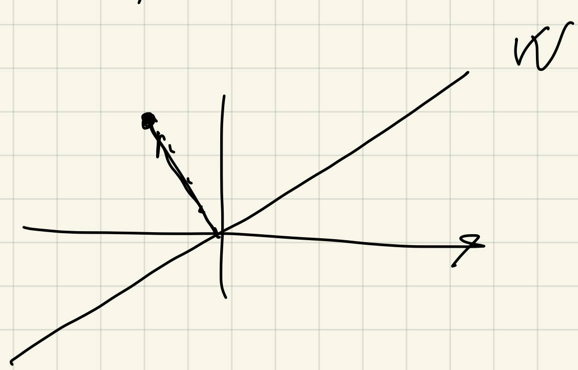
$$\Rightarrow \quad \left\| \left(1-\chi_{[-N_\varepsilon,N_\varepsilon]}\right) \hat{k}(n) \hat{f} \right\|_{\ell^2(\mathbb{Z}^d)} \leq$$

$$\underbrace{\left\| \left(1-\chi_{[-N_\varepsilon,N_\varepsilon]}\right) \hat{k}(n) \right\|_{\ell^\infty(\mathbb{Z}^d)}}_{\leq \varepsilon} \|\hat{f}\|_{\ell^2(\mathbb{Z}^d)}$$

__Lemma__ $W \subsetneq X \quad W$ closed, X B-space.

$\exists \quad \{w_n\}$ in $X \quad \|w_n\|_X = 1$ st

$$\operatorname{dist}(w_n, W) \longrightarrow 1$$



__Pf__ $v \in X \setminus W$, $\exists \; w_n \in W$ st

$$|v - w_n|_X \longrightarrow \mathrm{dist}(v, W) > 0$$

$$v_n = \frac{v - w_n}{|v - w_n|_X}$$

$$\mathrm{dist}(v_n, W) \leq \mathrm{dist}(v_n, 0) = |v_n|_X = 1$$

Claim $\lim_{n \to +\infty} \mathrm{dist}(v_n, W) = 1$

Suppose false

$$S := \liminf_{n \to +\infty} \mathrm{dist}(v_n, W) < 1$$

$S < a < 1$, there is a subsequence n

$\tilde{u}_n \in W$ s.t. $v_n = \frac{v - w_n}{\|v - w_n\|}$

$$\cancel{\mathrm{dist}} \ \|v_n - \tilde{u}_n\| < a$$

$$\|v - w_n - \|v - w_n\| \tilde{u}_n\| < a \|v - w_n\| \xrightarrow{n \to +\infty} a \,\mathrm{dist}(v, W)$$

$$u_n = w_n + \|v - w_n\| \tilde{u}_n \in W$$

$a < 1 \quad \exists\, \varepsilon > 0$

$$\|v - u_n\| < a \|v - w_n\| \longrightarrow a \,\mathrm{dist}(v, W) < \mathrm{dist}(v, W) - \varepsilon$$

So for $n \gg$

$\mathrm{dist}(v,W) \leq \| v - u_n \| < \mathrm{dist}(v,W) - \frac{\varepsilon}{2}$

$$\Rightarrow \quad S = 1$$

Corollary V B sp with $\overline{D_V(0,1)}$ compact

$$\Rightarrow \quad \dim V < \infty$$

Pf Suppose false and $\dim V = \infty$

It is possible to define a strictly increasing family of closed vector spaces E_n

and it is possible to choose $\forall n$

$$a \quad u_n \in E_n \setminus E_{n-1} \quad \text{st}$$

$$\mathrm{dist}(u_n, E_{n-1}) \geq \frac{1}{2}. \qquad \|u_n\| = 1$$

Then $\quad \|u_n - u_m\| \geq \frac{1}{2} \quad \cancel{\times}$

$$n > m \qquad u_m \in E_m \subseteq E_{n-1}$$

If $\{u_n\}$ is a sequence in $\overline{D_V(0,1)}$

there must exist a convergent subsequence

but for $\cancel{\times}$ this is impossible. We get a contradiction

Remark $T \in \mathcal{L}(\mathbb{R}^d)$

$$R(T) = \mathbb{R}^d \iff \ker T = 0 \quad \times$$

$1 \leq p \leq \infty$

$\ell^p(\mathbb{N})$

$$(x_1, x_2, \dots) \longrightarrow (0, x_1, x_2, \dots\dots)$$

has 0 kernel but obviously it is not surjective.

X B-space

$T = A + K$ K compact and

A an isomorphism then

$$R(T) = X \iff \ker T = 0$$

$$T = 1 - K$$

Theorem X B-space K compact operator

$T = 1 - K$

1) $\dim \ker T < \infty$

2) $R(T) = (\ker T^*)^\perp$

3) $\ker T = 0 \iff R(T) = X$

4) $\dim \ker T = \dim \ker T^*$

Remark $\quad K \in K(X)$

$$\sigma(K) = \{ \lambda \in \mathbb{C} : \lambda - K \text{ is not an isomorphism} \}$$

$0 \in \sigma(K)$ and if $\lambda \in \sigma(K) \setminus \{0\}$

then the algebraic dimension of λ is finite

$$N_g(\lambda - K) = \bigcup_{n=1}^{\infty} \ker (\lambda - K)^n$$

$$\dim N_g(\lambda - K) < +\infty$$

$$X = X_0 \oplus \bigoplus_{\lambda \in \sigma(K)\setminus\{0\}} N_g(\lambda - K)$$

$$\sigma\left(K|_{X_0}\right) = \{0\}$$

$N_g(\lambda - K)$ is finite dimensional space

$$\sigma\left(K|_{N_g(\lambda-K)}\right) = \{\lambda\}$$

$$K = \begin{pmatrix} \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} & \\ & \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \ddots & 1 \\ & & \lambda \end{pmatrix} \\ & & \ddots \end{pmatrix}$$

$sp\{e_1, \dots\dots, e_n\} = \cancel{N_\delta(\lambda - K)}$

$$K \rightsquigarrow \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

$T = 1 - K$

$$\begin{pmatrix} 1-\lambda & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & 1-\lambda \end{pmatrix} \qquad \text{if } \lambda \neq 1$$

this is an isomorfism $sp\{e_1, \dots, e_n\} \circlearrowleft$

$R(T)$ is the whole space

$\ker T = 0$

$\lambda = 1$

$$T \qquad \begin{pmatrix} 0 & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & 0 \end{pmatrix} \qquad \begin{aligned} Te_1 &= 0 \\ Te_2 &= -e_1 \\ &\vdots \\ Te_n &= -e_{n-1} \end{aligned}$$

$\ker T = sp\{e_1\}$

$$X' \qquad \langle e_k^*, e_j \rangle_{X' \times X} = \delta_{kj}$$

$$X' = Sp\{e_1^*, \dots, e_n^*\} \oplus Sp^\perp\{e_1, \dots, e_n\}$$

T^*

$T^* e^*_k = - e^*_{k+1}$ $\qquad k < n$

$T^* e^*_n = 0$

$$\langle T^* e^*_k, e_j \rangle_{X' \times X} = \langle e^*_k, T e_j \rangle_{X' \times X}$$

$$= - \langle e^*_k, e_{j-1} \rangle_{X' \times X} = - \delta_{k, j-1}$$

$$\langle T^* e^*_k, e_j \rangle = - \delta_{k, j-1}$$

$$\langle \underbrace{T^* e^*_{j-1}}_{-e_j}, e_j \rangle = -1$$

$$T^* \quad \begin{pmatrix} 0 & & & \\ -1 & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & 0 \end{pmatrix}$$

$e^*_2 \longrightarrow -e_1$

$\vdots$

e

~~$R(T^*) =$~~

$\ker T^* = \{ e^*_n \}$

$R(T) = \{ e_1, \ldots, e_{n-1} \}$

$R(T) = \ker^{\perp} T^*$

$$R(T) = \left(k_0 T^{x}\right)^{+}$$