14 Dec

Def E, F B spaces, $T \in \mathcal{L}(E,F)$,

T is called a compact operator if it sends bounded set in E into relatively compact subsets of F

that is, when $T(D_E(0,1))$ is relatively compact in F.

Example $T: E \to F$ is a finite rank operator if $\dim R(T) < +\infty$

Finite rank operators between B-spaces are compact.

$T(B_E(0,1))$ is a bounded subset of $R(T)$ (is closed in F) and we know

$$\overline{T(D_E(0,1))} \subseteq R(T)$$

and d being closed and bounded in $R(T)$ is also compact.

Exercises

1) $\underline{E} \xrightarrow{T} F \xrightarrow{S} X$ all bounded

then, if T or S is compact, ~~the~~ the $S \circ T$ is compact

2) ~~I~~ if $T \in K(\underline{E}, F)$

and $x_n \rightharpoonup x$ in $\underline{E}$

then $Tx_n \to Tx$ in F strongly.

$$K \in L^q(\mathbb{R}^d), \qquad f \in L^p(\mathbb{R}^d)$$

$$Tf = K * f$$

$$|Tf|_{L^r(\mathbb{R}^d)} \leq |K|_{L^q} \, |f|_{L^p}$$

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$$

$$T \in \mathcal{L}\left(L^p(\mathbb{R}^d), L^r(\mathbb{R}^d)\right)$$

These are not compact $\quad 1 < p < \infty$

$\forall\, h \in \mathbb{R}^d \qquad \tau_h g(x) = g(x-h)$

$$\tau_h T = T \tau_h$$

$$\boxed{\tau_h\, \kappa * f\,(x) = \kappa * f(x-h) = = \int_{\mathbb{R}^d} \kappa(y)\, f(x-h-y)\, dy}$$

$$= \int_{\mathbb{R}^d} \kappa(x-h-y)\, f(y)\, dy$$

$$= \int_{\mathbb{R}^d} \kappa(y)\, \tau_h f(x-y)\, dy$$

$$= \kappa * \tau_h f$$

$$\Rightarrow \quad \tau_h \underbrace{\kappa * f}_{Tf} = \kappa * \tau_h f$$

$$\tau_h T = T \tau_h$$

$$\mathbb{R}^d \ni h_n \xrightarrow{n \to \infty} \infty$$

$$\Rightarrow \quad \tau_{h_n} f \longrightarrow 0 \qquad \forall\, f \in L^p(\mathbb{R}^d)$$

$$1 < p < \infty$$

If $T \ \ : L^p(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$, $Tf \neq 0$

was compact, since $|\tau_{h_n} f|_{L^p(\mathbb{R}^d)} = |f|_{L^p(\mathbb{R}^d)}$

then the set $\{T\tau_{h_n} f\}$ would be

relatively compact in $L^r(\mathbb{R}^d)$, thus there

would be a subsequence such that

$$T\tau_{h_{n_k}} f \xrightarrow{k \to +\infty} g \quad \text{in } L^r(\mathbb{R}^d)$$

$$|g|_{L^r(\mathbb{R}^d)} = \lim_{k \to +\infty} |T\tau_{h_{n_k}} f|_{L^r}$$

$$= \lim_{k \to +\infty} |\cancel{\tau_{h_{n_k}}} Tf|_{L^r} = |Tf|_{L^r} \neq 0$$

On the other hand

$$f_{n_k} \xrightarrow{k \to \infty} 0 \quad \text{in } L^p(\mathbb{R}^d)$$

$$\Rightarrow \quad Tf_{n_k} \longrightarrow 0 \quad \text{in } L^r(\mathbb{R}^d)$$

strongly

$$\Rightarrow \quad \lim_{k \to +\infty} |Tf_{n_k}|_{L^r} = 0$$

$$\boxed{\begin{array}{l} \mathbb{R}^d \qquad 0 < \gamma < d \\ 1 < p < q < +\infty \qquad \frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d} \\ Tf(x) = \int_{\mathbb{R}^d} |x-y|^{-\gamma} f(y)\, dy \end{array}}$$

$$|Tf|_{L^q(\mathbb{R}^d)} \leq C \, |f|_{L^p(\mathbb{R}^d)}$$

Hardy – Littlewood – Sobolev inequality

$$S_{p,\lambda} f(x) = \lambda^{\frac{p}{d}} f(\lambda x) \qquad \lambda \in \mathbb{R}_+$$

$$\boxed{T\, S_{p,\lambda} f = \lambda^{\gamma - d - \frac{d}{q} + \frac{d}{p}} \, S_{q\lambda} T f}$$

$$T S_{p\lambda} f(x) = \int_{\mathbb{R}^d} |x-y|^{-\gamma} \lambda^{\frac{d}{p}} f(\lambda y)\, dy$$

$$d(\lambda y)$$

$$= \lambda^{\frac{d}{p}} \lambda^{\gamma - d} \int_{\mathbb{R}^d} |\lambda x - \lambda y|^{-\gamma} f(\lambda y)\, \lambda^d \, dy$$

$$= \lambda^{\frac{d}{p}} \lambda^{\gamma-d} \int_{\mathbb{R}^d} |\lambda x - y|^{-\gamma} f(y)\, dy$$

$$= \lambda^{\frac{d}{p}} \lambda^{\gamma - d} \quad Tf(\lambda x)$$

$$= \lambda^{\frac{d}{p}} \lambda^{\gamma - d} \lambda^{-\frac{d}{q}} \underbrace{\lambda^{\frac{d}{q}} Tf(\lambda x)}_{\delta_{q\lambda} Tf}$$

$$T \delta_{p\lambda} = \lambda^{\frac{d}{p} + \gamma - d - \frac{d}{q}} \delta_{q\lambda} T$$

$$\frac{1}{p} = \frac{1}{q} + \frac{d-\gamma}{d}$$

$$T : L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d)$$

$$\frac{d}{p} + \gamma - d - \frac{d}{q} = 0$$

$$\frac{d}{p} = \frac{d}{q} + d - \gamma$$

$$\frac{d}{p} + \gamma - d - \frac{d}{q} = 0$$

$$|T \delta_{p\lambda} f|_{L^q} \leq C \, |\delta_{p\lambda} f|_{L^p} = C \, |f|_{L^p}$$

$$\lambda^{\overbrace{\frac{d}{p} + \gamma - d - \frac{d}{q}}^{\substack{<0 \\ >0}}} \cancel{\delta_{q\lambda}} Tf|_{L^q} \leq C \, |f|_{L^p}$$

~~Theor~~

Def A metric space (X, d) is "totally bounded" if $\forall\, \varepsilon > 0$ $\exists$ a finite covering of X with balls of radius ε.

Theor (X, d) is compact

$\Longleftrightarrow$ (X, d) is complete and and totally bounded

Lemma $K(E, F)$ is closed in $\mathcal{L}(E, F)$ for the uniform topology

Pf Consider $\overline{D_E(0, 1)}$ and let

$T \in \overline{K(E, F)} \subseteq \mathcal{L}(E, F)$

We need to show that $T\,\overline{D_E(0, 1)}$ is relatively compact in F.

It is enough to show that $T\,\overline{D_E(0, 1)}$ is totally bounded.

$\varepsilon > 0$. Let $S \in K(E,F)$ s.t.

$$|T - S|_{\mathcal{L}(E,F)} < \frac{\varepsilon}{2}$$

$S\,\overline{D_E(0,1)}$ is totally bounded

and so there exists $y_1, \dots, y_n \in F$

s.t. $S\,\overline{D_E(0,1)} \subset \bigcup_{j=1}^{n} D_F(y_j, \frac{\varepsilon}{2})$

$$\Rightarrow \quad T\,\overline{D_E(0,1)} \subset \bigcup_{j=1}^{n} D_F(y_j, \varepsilon)$$

$x \in \overline{D_E(0,1)} \Rightarrow \exists\, y_j$ s.t.

$$|Sx - y_j|_F < \frac{\varepsilon}{2}$$

$$|Tx - y_j|_F = |Tx - Sx + Sx - y_j|_F$$

$$\leq |Tx - Sx|_F + |Sx - y_j|_F$$

$$< |T - S|_{\mathcal{L}(E,F)} \underbrace{|x|_E}_{\leq 1} + \frac{\varepsilon}{2}$$

$$\leq |T-S|_{\mathcal{L}(E,F)} + \frac{\varepsilon}{2} < \varepsilon$$

$$\underbrace{|T-S|_{\mathcal{L}(E,F)}}_{< \frac{\varepsilon}{2}}$$

Theorem E, F $\quad T \in \mathcal{L}(E,F)$

$T^* \in \mathcal{L}(F', E')$

$$T \in K(E,F) \Longleftrightarrow T^* \in K(F', E')$$

Theorem If F is a Hilbert space then

$T \in K(E,F)$ is $T_n \longrightarrow T$ in $\mathcal{L}(E,F)$

where T_n are finite rank operators.

Pf $\quad T\,\overline{D_E(0,1)} \subseteq \bigcup_{j=1}^{N_\varepsilon} D_F(y_j, \varepsilon)$

$$G = \operatorname{sp}\{y_1, \dots, y_{N_\varepsilon}\}$$

$$P_G : F \longrightarrow G$$

$$P_G \circ T : E \longrightarrow G \qquad \dim G < +\infty$$

$$|x|_E \leq 1$$

$$|P_G T x - T x|_F < 2\varepsilon$$

We $\exists\, y_j$ s.t

$$|Tx - y_j|_F < \varepsilon$$

$$|P_G T x - T x - y_j + y_j|_F \leq$$

$$\leq |P_G T x - \overset{G}{\underset{\in}{}} y_j|_F + \underbrace{|Tx - y_j|_F}_{< \varepsilon}$$

$P_G y_j = y_j$

$$\leq |P_G (Tx - y_j)|_F + \varepsilon$$

$$\leq \underbrace{|Tx - y_j|_F}_{< \varepsilon} + \varepsilon < 2\varepsilon$$

$$\Rightarrow \quad |P_G T x - T x|_F < 2\varepsilon$$

$$|P_G T - T|_{\mathcal{L}(E,F)} \leq 2\varepsilon$$

$$\ell^p(\mathbb{Z}) = \left\{ \{x_n\}_{n\in\mathbb{Z}} : \sum_{n=-\infty}^{+\infty} |x_n|^p < +\infty \right\}$$

Let $\{a_n\}_{n\in\mathbb{Z}} \in c^0(\mathbb{Z}) = \left\{ \{y_n\}_{n\in\mathbb{Z}} : \lim_{n\to\infty} y_n = 0 \right\}$

$T_a : \ell^p(\mathbb{Z}) \circlearrowleft$

$$\{x_n\} \longrightarrow T\{x_n\} = \{a_n x_n\}$$

$|a_n|_{\ell^\infty(\mathbb{Z})} < +\infty \qquad |a_n| \leq |a_n|_{\ell^\infty}$

$$\left(\sum_{n\in\mathbb{Z}} |a_n x_n|^p \right)^{\frac{1}{p}} \leq \left(\sum_{n\in\mathbb{Z}} |a_\cdot|_{\ell^\infty}^p |x_n|^p \right)^{\frac{1}{p}}$$

$$= |a_\cdot|_{\ell^\infty} \; |x_\cdot|_{\ell^p}$$

$$|(T_a x)_\cdot|_{\ell^p} \leq |a_\cdot|_{\ell^\infty} \; |x_\cdot|_{\ell^p}$$

$$|T|_{\mathcal{L}(\ell^p)} \leq |a_\cdot|_{\ell^\infty}$$

Is T is compact.

$\lim_{n\to\infty} a_n = 0$ means

$\forall \varepsilon > 0 \quad \exists N_\varepsilon$ st $|n| > N_\varepsilon$

$$\Rightarrow \quad |a_n| < \varepsilon$$

$$T x_{\cdot} = \underbrace{\chi_{[-N_\varepsilon, N_\varepsilon]} T x_{\cdot}}_{\text{compact}} + \underbrace{\left(1 - \chi_{[-N_\varepsilon, N_\varepsilon]}\right) T x_{\cdot}}$$

$$\dim R\left(\chi_{[-N_\varepsilon, N_\varepsilon]} T\right) \le 2 N_\varepsilon + 1$$

$$\Big| \underbrace{\left(1 - \chi_{[-N_\varepsilon, N_\varepsilon]}\right) a_{\cdot}}_{b_{\cdot}} \; x_{\cdot} \Big|_{\ell^p} \le \underbrace{|b_{\cdot}|_{\ell^\infty}}_{<\varepsilon} |x_{\cdot}|$$

$$b_n = \left(1 - \chi_{[-N_\varepsilon, N_\varepsilon]}(n)\right) a_n$$

$$b_n = \begin{cases} 0 & \text{if} \quad |n| \le N_\varepsilon \\ a_n & \text{if} \quad |n| > N_\varepsilon \end{cases}$$

$$\Rightarrow \ |b_n| < \varepsilon$$

Ex If $\kappa \in L^1(\mathbb{T}^d)$ then

$$Tf = \kappa * f$$

$$T : L^1(\mathbb{T}^d) \to L^1(\mathbb{T}^d)$$

is a compact operator

d

$$\int_{[-\pi,\pi]^d} \kappa(x-y)\, f(y)\, dy \qquad L^1(\mathbb{T}^d) \quad L^1([-\pi,\pi]^d)$$

$$\Gamma \subset \mathbb{R}^d \qquad \Gamma = 2\pi \mathbb{Z}^d$$

$$f(x-\gamma) = f(x) \quad \forall \gamma \in \Gamma$$