19 dec

Theorem X B-space K compact operator

$T = 1 - K$

1) $\dim \ker T < \infty$

2) $R(T) = (\ker T^*)^\perp$

3) $\ker T = 0 \iff R(T) = X$

4) $\dim \ker T = \dim \ker T^*$

Pf 1) $N \doteq \ker T = \ker (1-K)$

$$(1-K) D_N(0,1) = 0$$

$$D_N(0,1) = K D_N(0,1) \subseteq K D_X(0,1)$$

$\Rightarrow \overline{D_N(0,1)}$ is compact in X and of N.

$\Rightarrow \dim N < +\infty$

2) $R(T) = (\ker T^*)^\perp$. We know that

$\overline{R(T)} = (\ker T^*)^\perp$. We need to show

$R(T) = \overline{R(T)}$.

Let $\{Tx_n\}$ be a sequence in $R(T)$

with $Tx_n \longrightarrow f$ in X

Suppose $\{x_n\}$ is a bounded sequence

$$x_n = \underbrace{Tx_n + Kx_n}_{} = (1-K)x_n + Kx_n$$

Then it is not restrictive to assume

that $Kx_n \longrightarrow g \in X$

$$\begin{array}{ccccc} x_n = & Tx_n & + & Kx_n & \longrightarrow f+g \in X \\ & \downarrow & & \downarrow & \\ & f & & g & \end{array}$$

$$Tx_n \longrightarrow Tf + Tg = f = T(f+g)$$

$$\Rightarrow f \in R(T)$$

Suppose now that $Tx_n \longrightarrow f$

with $\{x_n\}$ not bounded

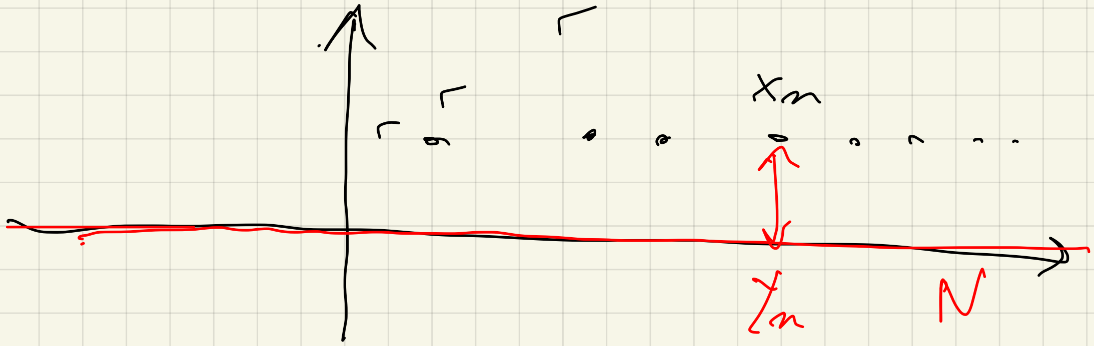
We claim

$\exists$ $\{y_n\}$ in $\ker T$ s.t. $\{x_n - y_n\}$ is bounded in X

$$T(x_n - y_n) = T x_n$$

$N = \ker T$ ~~but~~ is $\dim N < +\infty$

$$d_n \doteq \mathrm{dist}(x_n, N) = \inf_{y \in N} \|x_n - y\|$$



X

$$N \ni y \longrightarrow \|x_n - y\|$$

$$N \longrightarrow \mathbb{R}$$

$\exists$ a point of absolute minimum y_n

$d_n = \|x_n - y_n\|_X$. Suppose $d_n \nearrow +\infty$

$$f \leftarrow \frac{T x_n}{\|x_n - y_n\|_X} = T \underbrace{\frac{(x_n - y_n)}{\|x_n - y_n\|_X}}_{w_n} = w_n - \underbrace{K w_n}$$

$+\infty$ $\downarrow$ g $\downarrow$ g

~~Tx_n~~

0

$$\Longrightarrow \quad w_n \longrightarrow g$$

$$0 = g - Kg = Tg \qquad g \in N$$

$$w_n \longrightarrow g \in N$$

$$\underbrace{\operatorname{dist}(w_n, N)}_{\downarrow \atop 0} \leq \operatorname{dist}(w_n, g) \longrightarrow 0$$

$$\operatorname{dist}(w_n, N) = \operatorname{dist}\left(\frac{x_n - y_n}{d_n}, N\right) =$$

$$= \frac{1}{d_n} \operatorname{dist}(x_n - y_n, N) =$$

$$= \frac{1}{d_n} \overbrace{\operatorname{dist}(x_n, N)}^{d_n} = 1$$

We have a contradiction. So $\{d_n\}$ is bounded. $\qquad T = 1 - K$

3) $\ker T = 0 \Longleftrightarrow R(T) = X$

Let $\ker T = 0$ and suppose $X_1 = R(T) \subsetneq X$

$$X_{n+1} = T X_n \qquad\qquad T = 1 - K$$

$$T|_{X_n} \in \mathcal{L}(X_n) \qquad K|_{X_n} \in K(X_n)$$

$X_n := T X_{n-1}$ $\qquad$ $T X_n \subsetneq T X_{n-1}$

$X_{n+1} := T X_n$ $\qquad$ $X_{n+1} \subsetneq X_n$

Therefore $\{X_n\}$ is strictly decreasing

$\forall n \quad \exists \quad x_n \in X_n$ s.t. $\|x_n\| = 1$

$$\text{dist}\left(x_n, X_{n+1}\right) > \frac{1}{2}$$

$n > m$ (our ~~goal~~ goal is to ~~show~~ show that $\{Kx_n\}$ does not have convergent subsequences)

$$K\cancel{x}_m - K x_n = x_m - x_n - T x_m + T x_n$$

$$= x_m - \left(x_n + T x_m - T x_n\right) \in X_{m+1}$$

$x_n \in X_n$, $T x_m \in X_{m+1}$, $T x_n \in X_{n+1}$

$n > m$

$n \geq m+1$

$K x_m - K x_n$

$= x_m - x_{nm}$ $\qquad$ where $x_{nm} \in X_{m+1}$

$$\| K x_m - K x_n \| = \| x_m - x_{nm} \| \geq \text{dist}\left(x_m, X_{m+1}\right)$$

$$> \frac{1}{2}$$

$$\|Kx_m - Kx_n\| \geq \frac{1}{2} \quad \forall \quad n \neq m$$

$\Rightarrow \{Kx_n\}$ has no convergent subsequence

$\|x_n\| = 1$ But K is compact

$\ker T = 0$
$$\Longrightarrow R(T) = X$$

Let now $R(T) = X$ and let us prove $\ker T = 0$.

$$\ker T^* = R(T)^\perp \Rightarrow \ker T^* = 0$$

$$T^* : X' \longrightarrow X^* \qquad T^* = 1 - K^*$$

$$\Longrightarrow \quad R(T^*) = X'$$

$$\Longrightarrow \quad \ker T = R(T^*)^\perp$$

$$\Longrightarrow \quad \ker T = 0$$

<u>Thm</u> $K \in K(X)$ X B space $\dim X = +\infty$

1) $0 \in \sigma(K)$

2) $\lambda \in \sigma(K)$ $\lambda \neq 0 \Rightarrow$ λ is an eigenvalue

3) Either $\sigma(K)$ is finite or

$\sigma(K) \setminus \{0\}$ is a sequence convergent to 0.

4) $\lambda \in \sigma(K) \setminus \{0\}$ then algebraic dimension is finite.

<u>Pf</u> 1) If $0 \notin \sigma(K)$ then $K^{-1} \in \mathcal{L}(X)$

$1 = K\, K^{-1}$ would be compact

$\Rightarrow$ $D_X(0,1)$ is relatively compact in X

$\Rightarrow$ $\dim X < +\infty$.

2) $\lambda \in \sigma(K) \setminus \{0\}$. If λ is not an eigenvalue $\Rightarrow \ker(K - \lambda) = 0$

$\ker\left(\frac{K}{\lambda} - 1\right) = 0$ $\ker\left(1 - \frac{1}{\lambda}K\right) = 0$

$$R\left(1-\frac{1}{\lambda}K\right) = X$$

$\left(1-\frac{1}{\lambda}K\right) : X \longrightarrow X$. $\exists$ the inverse

$\left(1-\frac{1}{\lambda}K\right)^{-1} : X \to X$ and we show

it is bounded.

$$\operatorname{graph}\left(1-\frac{1}{\lambda}K\right)^{-1} = \left\{ \left(x, \underbrace{\left(1-\frac{1}{\lambda}K\right)^{-1}x}_{y}\right) : x \in X \right\}$$

$$y = \left(1-\frac{1}{\lambda}K\right)^{-1}x \iff x = \left(1-\frac{1}{\lambda}K\right)y$$

$$= \left\{ \left(\left(1-\frac{1}{\lambda}K\right)x, x\right) : x \in X \right\}$$

Notice that it is related

$$\operatorname{graph}\left(1-\frac{1}{\lambda}K\right) = \left\{ \left(x, \left(1-\frac{1}{\lambda}K\right)x\right) : x \in X \right\}$$

$\operatorname{graph}\left(1-\frac{1}{\lambda}K\right)^{-1}$ is closed in $X \times X$

$$\Rightarrow \left(1-\frac{1}{\lambda}K\right)^{-1} \in \mathcal{L}(X)$$

$$\iff (\lambda - K)^{-1} \in \mathcal{L}(X)$$

$$(K-\lambda)^{-1} \in \mathcal{L}(X) \Rightarrow \lambda \notin \sigma(K)$$

3) $\sigma(K)\setminus\{0\}$ is infinite.

$$\sigma(K) \subseteq \overline{D_{\mathbb{C}}(0, \|K\|_{\mathcal{L}(X)})}$$

Let $\{\lambda_n\}$ be a sequence of distinct elements of $\sigma(K)\setminus\{0\}$

Suppose $\quad \lambda_n \longrightarrow \lambda \neq 0$

$$K x_n = \lambda_n x_n \qquad \|x_n\|_X = 1$$

$$X_n = \operatorname{span}\{x_1, \dots, x_n\}$$

$$X_n \supsetneq X_{n-1}$$

$$\exists \; y_n \in X_n \qquad \operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$$

$$\|y_n\|_X = 1$$

$n > m$

$$\frac{K y_n}{\lambda_n} - \frac{K y_m}{\lambda_m} = y_n + \Big[\underset{\substack{\uparrow \\ X_m}}{-y_m} + \underset{\substack{\uparrow \\ X_{n-1}}}{\frac{(K-\lambda_n)y_n}{\lambda_n}} - \underset{\substack{\uparrow \\ X_{m-1}}}{\frac{(K-\lambda_m)y_m}{\lambda_m}} \Big]$$

$$= y_n - \underset{\substack{\uparrow \\ X_{n-1}}}{z_{mn}}$$

$$(K-\lambda_n)y_n = \frac{\lambda_n}{\lambda_m} K y_m + \lambda_n z_{nm} \in X_{n-1}$$

$$\left\| \frac{K y_n}{\lambda_n} - \frac{K y_m}{\lambda_m} \right\|_X = \| y_n - \underset{\substack{\cap \\ X_{n-1}}}{z_{mn}} \|_X \geq$$

$$\geq \operatorname{dist}(y_n, X_{n-1}) \geq \frac{1}{2}$$

$$\left\| \frac{K y_n}{\lambda_n} - K \frac{y_m}{\lambda_m} \right\| > \frac{1}{2} \quad \forall\ n \neq m$$

$\Rightarrow$ $K \frac{y_n}{\lambda_n}$ does not have convergent subsequences

$$K \frac{y_n}{\lambda_n} = \frac{1}{\lambda} K\, y_n + \underbrace{\left(\frac{1}{\lambda_n} - \frac{1}{\lambda}\right) K y_n}_{\downarrow\ 0}$$

There must be ~~a~~ convergent subsequence

H $\quad \{f_n\}$ $\quad$ a basis

$a_n \searrow 0$ $\quad$ strictly decreasing

$$A = \sum_{n=1}^{\infty} a_n (\cdot, f_n)_H f_{n+1}$$

$\ker A = 0$ $\qquad \sigma(A) = \{0\}$

By induction

$$A^m f = \sum_{n=m}^{+\infty} a_n a_{n-1} \cdots a_{n-m+1} (f, f_{n-m+1}) f_{n+1}$$

$$\|A^m\|^{\frac{1}{m}} = (a_m a_{m-1} \cdots a_{1})^{\frac{1}{m}}$$

$\lambda \in A^m \qquad |\lambda| \le \|A^m\|^{\frac{1}{m}} \quad \forall m$

$$\le \frac{a_1 + \cdots + a_m}{m} \xrightarrow{m \to +\infty} 0$$