AG 3 - Christmas exercises

Valentina Beorchia

December 22, 2022

1. Let $X \subset \mathbb{P}^n$ with $n \geq 2$ be a reducible hypersurface. Prove that
$$\dim X_{sing} \geq n-2.$$

2. Let $0 < m \leq n$ and consider the hyperquadric $Q_m = V_P(x_0^2 + \cdots + x_m^2) \subset \mathbb{P}^n$. Prove that Q_m is reducible if and only if $m = 1$.

3. Consider the Grassmannian $\mathbb{G}(1,3) \subset \mathbb{P}^5_{\mathbb{C}}$ of lines in $\mathbb{P}^3$. Prove that for a fixed projective subspace $\Lambda \subset \mathbb{P}^3$, $\Lambda \cong \mathbb{P}^2$, the subset $\Gamma_\Lambda := \{l \in \mathbb{G}(1,3) | l \subset \Lambda\}$ is closed and compute its dimension.

4. Consider the Grassmannian $\mathbb{G}(1,3) \subset \mathbb{P}^5_{\mathbb{C}}$ of lines in $\mathbb{P}^3$. Prove that if $Q = V_P(x_0x_3 - x_1x_2)$, then $\Gamma_Q := \{l \in \mathbb{G}(1,3) | l \subset Q\}$ is closed and compute its dimension.

5. **Definition**: **Gauss Map and Tangent Variety** Let $X \subseteq \mathbb{P}^n$ be a smooth projective variety of dimension k, and let $\mathbb{T}_pX$ be the projective embedded tangent space to X at $p \in X$.

   The Gauss Map is the morphism
$$G : X \to \mathbb{G}(k,n), \quad p \to \mathbb{T}_pX.$$

   Moreover, the *Tangent variety* $T(X)$ is defined as
$$T(X) := \bigcup_{\Lambda \in G(X)} \Lambda \subseteq \mathbb{P}^n.$$

   Prove that $T(X)$ is a projecive variety of dimension
$$\dim T(X) \leq 2 \dim X.$$

6. Let $X, Y \subset \mathbb{P}^n$ be disjoint projective varieties. Show that the *join of X and Y*:
$$J(X,Y) = \bigcup_{L \in W} L \subseteq \mathbb{P}^n,$$

   where
$$W = \{L \in \mathbb{G}(1,n) \ : \ L \cap X \neq \emptyset, \ L \cap Y \neq \emptyset\}$$

   is closed in $\mathbb{P}^n$.

7. Consider the projective surface $X = V_P(x_1x_2^2 - x_0x_3^2) \subset \mathbb{P}^3$. Determine $Z = X_{Sing}$ and find equation for the blow up $\widetilde{X} = \mathrm{Bl}_Z X$. Is $\widetilde{X}$ smooth?

8. Prove that the locus of surfaces of degree 4 in $\mathbb{P}^3$ containing a line is a hypersurface in $\mathbb{P}^{34}$.