21 Dic $\quad K = \mathbb{R}, \mathbb{C}$

__Def__ $\quad H \qquad B : H \times H \longrightarrow K$

bilinear or (if $K = \mathbb{C}$) sesquilinear

is bounded if $\exists \quad \gamma \in \mathbb{R}_+$

s.t. $\quad |B(x,y)| \leq \gamma \, \|x\|_H \, \|y\|_H \quad \forall x,y \in H$

and is coercive if $\exists \quad \delta \in \mathbb{R}_+$ s.t.

$$|B(x,x)| \geq \delta \, \|x\|_H^2 \qquad \forall x \in H.$$

__Th__ Suppose B is bounded and coercive.

$\exists \quad S \in \mathcal{L}(H)$ with $S^{-1} \in \mathcal{L}(H)$

s.t. $\quad (x,y)_H = B(x, Sy)$

$$\|S\|_{\mathcal{L}(H)} \leq \delta^{-1}$$

$$\|S^{-1}\|_{\mathcal{L}(H)} \leq \gamma$$

If B is symmetric (hermitian)

then S is selfadjoint

Pf Let

$$D := \{ y \in H : \exists\, y^* \in H \text{ s.t. } (x, y)_H = B(x, y^*) \ \forall x \in H \}$$

$D \ni 0 \qquad 0^* = 0$

$\forall y \ \exists!\ y^*$ for which the above is true

There is no $\cancel{0}^* = z \neq 0$

$$0 = (x, 0)_H = B(\cancel{x}, z) \qquad \forall x \in H$$

$$0 = (z, 0)_H = B(z, z)$$

$$|B(z,z)| \geq \delta\, |z|_H^2 > 0 \quad \text{if } z \neq 0$$

$$D \ni y \longrightarrow Sy = y^*$$

$S \in \mathcal{L}(D, H) \qquad \|S\|_{\mathcal{L}(D,H)} \leq \delta^{-1}$

$$\delta\, |Sy|_H^2 \leq |B(Sy, Sy)| = |(Sy, y)_H| \leq$$

$$\leq |Sy|_H\, |y|_H$$

$$\delta\, |Sy|_H \le |y|_H$$

$$|Sy|_H \le \delta^{-1} |y|_H \qquad \forall y \in D$$

$$\Rightarrow \quad |S|_{\mathcal{L}(D,H)} \le \delta^{-1}$$

$$\bar{S} \in \mathcal{L}(\bar{D}, H)$$

$$|\bar{S}|_{\mathcal{L}(\bar{D},H)} \le \delta^{-1}$$

$$D = \bar{D} = H$$

$$D = \bar{D}$$

$$D \ni y_n \xrightarrow{n \to +\infty} z \in \bar{D} \subseteq H$$

$$S y_n \xrightarrow{n\to\infty} \bar{S} z$$

$$(x,z)_H = \lim_{n\to+\infty} (x, y_n)_H =$$

$$= \lim_{n\to+\infty} B(x, S y_n) = B(x, \bar{S} z)$$

$$(x,z)_H = B(x, \bar{S} z) \qquad \forall x \in H$$

$$\bar{S} z = z^* \Rightarrow z \in D$$

with $\quad Sz = \bar{S}z$

$D = \bar{D}$ $\qquad D = H$

Suppose $\quad D \subsetneq H$

$0 \neq w_0 \in D^{\perp}$

$\{x \longrightarrow B(x, w_0)\} \in H'$

$|B(x, w_0)| \leq \gamma \, |x|_H \, |w_0|_H$

$\Longrightarrow \quad \exists \; w \in H \;$ s.t.

$B(x, w_0) = (x, w)_H \qquad \forall \; x \in H$

$w \in D$ with $\quad w_0 = Sw \quad (SD = H)$

$(w, w_0)_H = 0$

$0 < \delta |w_0|_H^2 \leq |B(w_0, \overset{Sw}{w_0})| = |(w_0, w)_H| = 0$

$0 < 0$

$D = H$

$S : H \longrightarrow H \qquad |S|_{\mathcal{L}(D,H)} \leq \delta^{-1}$

$$|S|_{\mathcal{L}(H)} \leq \delta^{-1}$$

$S : H \to H$ $\qquad R(S) = H$

S is bounded injective

$\ker S = 0$ $\qquad y \neq 0 \quad Sy = 0$

$$0 = B(x, \underbrace{Sy}_{0}) = (x, y)_H \qquad \forall\, x \in H$$

$$(x, y)_H = 0 \qquad \forall\, x \in H$$

$x = y$ $\qquad (y, y)_H = |y|_H^2 = 0 \implies y = 0$

$$|S^{-1}|_{\mathcal{L}(H)} \leq \gamma$$

$$B(x, Sy) = (x, y)_H \qquad \forall\, x, y$$

$$B(x, y) = (x, S^{-1}y)_H \qquad \forall\, x, y)$$

$$|(x, S^{-1}y)_H| \leq |B(x, y)| \leq$$

$$\leq \gamma\, |x|_H\, |y|_H \qquad \forall\, x \in H$$

$$\implies \|S^{-1}y\|_H \leq \gamma\, |y|_H \qquad \forall\, y \in H$$

$$\Longrightarrow \quad \|S^{-1}\|_{\mathcal{L}(H)} \leq \gamma$$

<u>Corollary</u> $f' \in H'$ and consider the problem of finding a $u \in H$ s.t.

$$\boxed{B(v,u) = \langle v, f' \rangle_{H \times H'} \qquad \forall v \in H} \quad (1)$$

Then $\exists !\, u$ which satisfies (1) and is given by $u = Sf$ where $f \in H$ is s.t. $(v, f)_H = \langle v, f' \rangle_{H \times H'}$ $\forall v \in H$

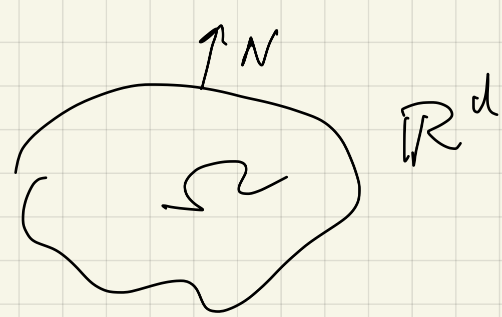
<u>Pf</u>

$$(v, f)_H = \langle v, f' \rangle_{H \times H'} \qquad \forall v \in H$$

$$= B(v, Sf) \qquad \cancel{\forall v \in}$$

$$B(v, Sf) = \langle v, f' \rangle_{H \times H'} \qquad \forall v \in H$$

$$u = Sf$$



$$\begin{cases} -\partial_i\left(a_{ij}\,\partial_j u\right) = f \\ N_i a_{ij}\partial_j u \,\big|_{\partial\Omega} = 0 \\ \cancel{N\cdot\nabla u}\,\big|_{\partial\Omega} = 0 \end{cases}$$

$$\begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & & \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{pmatrix}$$

$\forall v \in H^1(\Omega)$

$$\boxed{\int_\Omega (-)\,\partial_i\left(a_{ij}\,\partial_j u\right) v \, dx = \int f\, v\, dx}$$

$$-\partial_i\left(a_{ij}\,\partial_j u\right) v = -\partial_i\left(a_{ij}\,\partial_j u\, v\right) + a_{ij}\,\partial_j u\, \partial_i v$$

$$\int_\Omega (-1)\,\partial_i\left(a_{ij}\,\partial_j u\right) v\, dx =$$

$$= -\int_\Omega \partial_i\left(a_{ij}\,\partial_j u\, v\right) dx + \int_\Omega a_{ij}\,\partial_i v\, \partial_j u\, dx$$

$$= \underbrace{-\int_{\partial\Omega} N_i\, a_{ij}\,\partial_j u\, v}_{0} + \quad ''$$

$$\boxed{\int_\Omega a_{ij}\, \partial_i v\, \partial_j u\, dx = \int vf} \quad \forall v \in H^1(\Omega)$$

$B(u,v)$

Th (Spectral dec of a self adj. compact operator)

$H \qquad T \in K(H) \qquad T^* = T$

Then $\exists$ an orthonormal basis of H formed by eigenv. of T

$$H = \overline{\bigoplus_{\lambda \in \sigma(T)} \ker(T-\lambda)}$$

$$x \rightsquigarrow (x_\lambda)_{\lambda \in \sigma(T)}$$

$$Tx \to (\lambda x_\lambda)_{\lambda \in \sigma(T)}$$

$$\mathbb{T}^d \xrightarrow{f} \mathbb{C} \qquad \mathbb{Z}^d \xrightarrow{\hat{f}} \mathbb{C}$$

$$f(x) = \sum \hat{f}(n) \frac{e^{in\cdot x}}{(2\pi)^{\frac{d}{2}}}$$

$$\langle \xi \rangle = \sqrt{1 + |\xi|^2}$$

$s \in \mathbb{R}$ $\qquad H^s(\mathbb{T}^d)$ Sobolev space

$$\| f \|^2_{H^s} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} \, |\hat{f}(n)|^2$$

$$= \| \langle n \rangle^s \hat{f} \|_{\ell^2(\mathbb{Z}^d)}$$

$m \in \mathbb{N}$ $\qquad s = m$

$$||| f |||^2 = \sum_{0 \le j \le N} \| \nabla^j f \|^2_{L^2(\mathbb{T}^d)}$$

$s_1 > s_2$

$$H^{s_1}(\mathbb{T}^d) \hookrightarrow H^{s_2}(\mathbb{T}^d) \quad \text{it is compact}$$

$$s > \frac{d}{2}$$

$$H^s(\mathbb{T}^d) \hookrightarrow \overset{L^\infty}{C^0}(\mathbb{T}^d)$$

$$f(x) = \sum \hat{f}(n) \frac{e^{in \cdot x}}{(2\pi)^{\frac{d}{2}}}$$

$$|f(x)| \le \sum_{n \in \mathbb{Z}^d} \left(|\hat{f}(n)| \langle n \rangle^s \right) \left(\langle n \rangle^{-s} \right)$$

$$\leq \left(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{-2s} \right)^{\frac{1}{2}} \underbrace{\left(\sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2 \right)^{\frac{1}{2}}}_{\|f\|_{H^s}}$$

$2s > d$ $\qquad C_s$

$$|f(x)| \leq C_s \|f\|_{H^s}$$

$$0 < s < \frac{d}{2}$$

$$H^s(\mathbb{T}^d) \hookrightarrow L^{p_s}(\mathbb{T}^d)$$

$\mathbb{R}^d$ $\qquad \mathbb{R}^d$

is not compact

$$\frac{1}{p_s} = \frac{1}{2} - \frac{s}{d}.$$

$$2 < p_s < +\infty$$

$H^1(\mathbb{T}^d) \qquad V \in C^0(\mathbb{T}^d, [0, +\infty))$

$$B(u,v) = (\nabla u, \nabla v)_{L^2(\mathbb{T}^d)} + (Vu, v)_{L^2(\mathbb{T}^d)}$$

$$B : H^1 \times H^1 \longrightarrow \mathbb{C}$$

is bounded, coercive and Hermitian

$\forall f \in H^{-1}$ $\qquad H^1$

$\forall v \in H^1$ $\quad B(u,v) = (f,v)_{L^2}$ $\qquad H^1 \qquad H^1$

$(-\Delta u, v)_{L^2} + (Vu, v)_{L^2} = (f,v)_{L^2}$ $\quad H^1 \times H^{-1}$

$((-\Delta + V)u, v)_{L^2} = (f,v)_{L^2}$ $\quad \forall v$ $\quad \langle \,,\, \rangle_{H^1 \times H^{-1}} = (\,,\,)_{L^2}$

$$\boxed{(-\Delta + V)u = f}$$

$B(S\cancel{f}f, v) = (f,v)$

$u = Sf$

$S = (-\Delta + V)^{-1}$

S selfadjoint and compact in $L^2(\mathbb{T}^d)$

$\exists$ $\{e_n\}$ orthonormal basis in $L^2(\mathbb{T}^d)$

and $\{\mu_n\}$ in $\mathbb{R} \setminus \{0\}$

$\Rightarrow$ $\quad S e_n = \mu_n e_n \quad \forall n$

$$(-\Delta+V)^{-1}e_n = \mu_n e_n$$

$$\lambda_n = \frac{1}{\mu_n}$$

$$\frac{1}{\mu_n} e_n = (-\Delta+V) e_n$$

$$(-\Delta+V) e_n = \lambda_n e_n$$

$$e_n$$

$$-\Delta \qquad \frac{e^{inx}}{(2\pi)^{\frac{d}{2}}} \qquad L^2(\mathbb{T}^d)$$

$$f(x) = \sum_{n\in\mathbb{Z}^d} \hat{f}(n) \frac{e^{inx}}{(2\pi)^{\frac{d}{2}}}$$