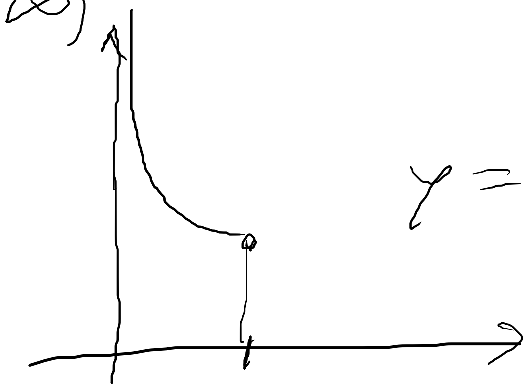
20 dicembre Mattina. Integrali Impropri

$$\int_0^{+\infty} \frac{1}{1+x^4}\,dx$$ Integrale su $[0,+\infty)$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$ Integrale su $[0,1]$ di



$y = \frac{1}{\sqrt{x}}$

$$\int_{\mathbb{R}} e^{-x^2}\,dx$$ Integrale su $\mathbb{R}$

(e^{-x^2} è la più famosa funzione di cui non si ha una formula esplicita per le primitive.)

Def Sia $f \in L_{loc}([a,b))$ con $a \in \mathbb{R}$ e $b \in (a, +\infty]$. Diciamo che f è integrabile (o sommabile) in $[a,b)$ se esiste ed è finito il limite

$$\lim_{R \to b^-} \int_a^R f(x)\,dx = \int_a^b f(x)\,dx$$

e scriviamo $f \in L([a,b))$

Es Le funzioni x^{-p} sono integrabili in $[1,+\infty)$ se e solo se $p>1$.

Notare che $x^{-p} \in C^\infty([1,+\infty)) \Rightarrow x^{-p} \in L_{loc}([1,+\infty))$ $\forall\, p \in \mathbb{R}$.

$$\lim_{R \to +\infty} \int_1^R x^{-p}\,dx$$

Sia $p \neq 1$ $\quad \int_1^R x^{-p}\,dx = \left.\frac{x^{1-p}}{1-p}\right]_1^R =$

$$= \frac{R^{1-p}}{1-p} - \frac{1}{1-p}$$

$\int_1^{+\infty} x^{-p}\,dx$

$\parallel$

Se $p>1$, allora $1-p<0$ e pertanto

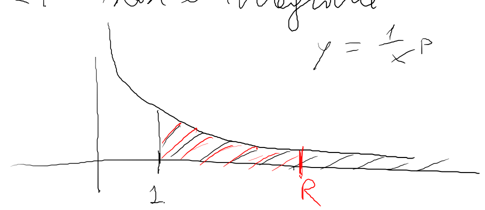
$$\int_1^R x^{-p}\,dx = \frac{R^{1-p}}{1-p} - \frac{1}{1-p} \xrightarrow{R \to +\infty} -\frac{1}{1-p} = \frac{1}{p-1} > 0$$

Se $p<1$ allora $1-p>0$

$$\int_1^R x^{-p}\,dx = \frac{R^{1-p}}{1-p} - \frac{1}{1-p} \xrightarrow{R \to +\infty} +\infty$$

$\int_1^{+\infty} x^{-p}\,dx = +\infty$
$p<1$

x^{-p} per $p<1$ non è integrabile in $[1,+\infty)$

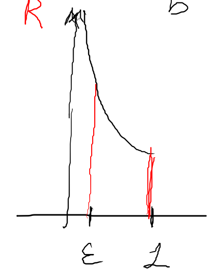


$p=1$ $\quad \int_1^R \frac{1}{x}\,dx = \left.\lg x\right]_1^R = \lg R \xrightarrow{R \to +\infty} +\infty$

$\frac{1}{x} \notin L([1,+\infty))$

$f \in L_{loc}([a,b))$ risulta $f \in L([a,b))$ se esiste finito

$$\lim_{R\to b^-} \int_a^R f(x)\,dx$$
$$=: \int_a^b f(x)\,dx$$



Es x^{-p} $(0,1]$ per $p>0$

Risulta x^{-p} integrabile in $(0,1]$ se

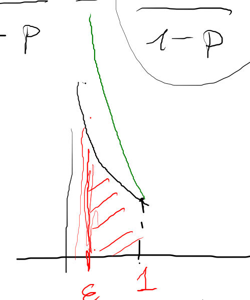
$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{-p}\,dx$$ esiste ed è finito

Risulta $x^{-p} \in L((0,1])$ se $p<1$ mentre

$x^{-p} \notin L((0,1])$ se $p \geq 1$.

$p<1$
$$\int_\varepsilon^1 x^{-p}\,dx = \left.\frac{x^{1-p}}{1-p}\right]_\varepsilon^1 = \frac{1}{1-p} - \frac{\varepsilon^{\overbrace{1-p}^{<0}}}{1-p} \xrightarrow{\varepsilon\to 0^+} 0$$

$\Longrightarrow$ per $p<1$
$$\lim_{\varepsilon\to 0^+} \int_\varepsilon^1 x^{-p}\,dx = \frac{1}{1-p}$$



Sia $p>1$
$$\int_\varepsilon^1 x^{-p}\,dx = \frac{1}{1-p} - \frac{\varepsilon^{\overbrace{1-p}^{<0}}}{1-p} \xrightarrow{\varepsilon\to 0^+} \frac{1}{1-p} - \frac{\overbrace{+\infty}^{+\infty}}{1-p} = +\infty$$

Sia $p=1$
$$\int_\varepsilon^1 \frac{1}{x}\,dx = \lg x\Big]_\varepsilon^1 = -\lg\varepsilon \xrightarrow{\varepsilon\to 0^+} +\infty$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \int_0^1 x^{-\frac{1}{2}}\,dx = \frac{1}{1-\frac{1}{2}} = 2$$

$$\left(\int_{\mathbb{R}} e^{-x^2}\,dx\right)^2 = \int_{\mathbb{R}} e^{-x^2}\,dx \int_{\mathbb{R}} e^{-y^2}\,dy =$$

$$r^2 = x^2 + y^2$$

$$= \int_{\mathbb{R}\times\mathbb{R}} e^{-x^2-y^2}\,dx\,dy$$

$$x = r\cos\vartheta$$

$$y = r\sin\vartheta$$

$$= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\, r\,dr\,d\vartheta \qquad dx\,dy = r\,dr\,d\vartheta$$

$$= \underbrace{\int_0^{2\pi} d\vartheta}_{2\pi} \quad \underbrace{\int_0^{+\infty} r e^{-r^2}\,dr}_{\frac{1}{2}} = \pi$$

$$\int_0^{+\infty} r e^{-r^2}\,dr = \lim_{R\to+\infty} \int_0^R \cancel{r}\, e^{-r^2}\,dr = \frac{1}{2}$$

$$\int_0^R r\, e^{-r^2}\,dr = \qquad u = r^2 \qquad du = 2r\,dr$$

$$= \frac{1}{2}\int_0^{R^2} e^{-u}\,du = \frac{1}{2} \qquad -e^{-u}\Big]_0^{R^2} = \frac{1}{2}\left(1 - e^{-R^2}\right) \xrightarrow{R\to+\infty}$$

$$= 1$$

$$\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

1) $f, g \in L([a,b))$ e se $\lambda, \mu \in \mathbb{R}$, allora

$\lambda f + \mu g \in L([a,b))$ con

$$\int_a^b (\lambda f + \mu g)\, dx = \lambda \int_a^b f\, dx + \mu \int_a^b g\, dx$$

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$$

2) $f, g \in L([a,b))$ con $f(x) \le g(x) \quad \forall x \in [a,b)$

$$\Longrightarrow \quad \int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$$

3) Se $f \in L_{loc}([a,b))$ $\forall\, c \in [a,b)$ allora $f \in L([a,b)) \Longleftrightarrow f \in L([c,b))$

e si ha

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Teor (Aut - aut) $f \in L_{loc}([a,b))$ $f(x) \geq 0$ $\forall x \in [a,b)$

Allora $\lim_{R \to b^-} \int_a^R f(x)\,dx$ esiste ed è finito

(nel qual caso $f \in L([a,b))$) oppure $= +\infty$.

<u>Dim</u> Poniamo $F(x) = \int_a^x f(t)\,dt$. Risulta che

$F(x)$ è crescente. ~~Infatti~~ se $a \leq x_1 < x_2$

$$F(x_2) = \int_a^{x_2} f(t)\,dt = \int_a^{x_1} f(t)\,dt + \underbrace{\int_{x_1}^{x_2} \underbrace{f(t)}_{\geq 0}\,dt}_{\geq 0}$$

$$\geq \int_a^{x_1} f(t)\,dt = F(x_1)$$

$\Rightarrow$ $F(x_2) \geq F(x_1) \geq F(a) = 0$ se $x_2 > x_1 \geq a$

$\Rightarrow$ $\lim_{x \to b^-} F(x) = \sup\{F(x) : a \leq x < b\} \in [0, +\infty]$

$\lim_{x \to b^-} \int_a^x f(t)\,dt$

Esercizio 1 Sia f integrabile per Riemann in $[a,b] \subseteq \mathbb{R}$

Dimostrare allora che $f \in L([a,b))$ e che integrale di Riemann e integrale improprio coincidono

Esercizio 2 Sia f limitata in $[a,b]$ ed integrabile in senso improprio in $[a,b)$. Allora f è integrabile per Riemann in $[a,b]$.