20 dicembre Pomeriggio

**Teor (Confronto)** Siano $f, g \in L_{loc}([a,b))$ con $0 \le f(x) \le g(x)$ $\forall x$. Allora se $g \in L([a,b))$ si ha $f \in L([a,b))$

**Dim** $f(x) \le g(x)$ $\forall x \Rightarrow \int_a^R f(x)\,dx \le \int_a^R g(x)\,dx$ $\forall R \in (a,b)$. Sappiamo che Aut-at che

$\lim_{R\to b^-} \int_a^R f(x)\,dx$ esiste in $[0,+\infty]$

$\le \lim_{R\to b^-} \int_a^R g(x)\,dx \in [0,+\infty)$

$$\Longrightarrow \lim_{R\to b^-} \int_a^R f(x)\,dx \in [0,+\infty)$$

$$\Longleftrightarrow f \in L([a,b)) \qquad \square$$

**Osservazione** $f, g \in L_{loc}([a,b))$ e $0 \le f(x) \le g(x)$

$g \in L([a,b)) \Rightarrow f \in L([a,b))$ $\qquad P_1 \Rightarrow P_2$

$f \notin L([a,b)) \Rightarrow g \notin L([a,b))$ $\qquad \cancel{P_2} \Rightarrow \cancel{P_1}$

Es. $\int_0^{+\infty} \frac{x^3}{1+x^4}\,dx$ è sommabile?

La domanda è equivalente alla seguente:

$\int_1^{+\infty} \frac{x^3}{1+x^4}\,dx$ è sommabile?

$x \ge 1$

$x^4 \ge 1$

$x^{-4} \le 1$

$$\frac{x^3}{1+x^4} = \frac{\cancel{x^3}}{x^{\cancel{4}\,1}} \frac{1}{1+x^{-4}}$$

$$= \frac{1}{x} \frac{1}{1+x^{-4}} \ge \frac{1}{x} \frac{1}{2}$$

$$\frac{x^3}{1+x^4} \ge \frac{1}{2} \frac{1}{x} \quad \forall x \ge 1$$

$$\frac{1}{2} \frac{1}{x} \notin L([1,+\infty)) \Longrightarrow \frac{x^3}{1+x^4} \notin L([1,+\infty))$$

Teor $f, g \in L_{loc}([a,b))$, $f(x) \geq 0 \ \forall x$,

$g(x) > 0 \ \forall x$.

Supponiamo che $\lim_{x \to b^-} \frac{f(x)}{g(x)} = L \ (\geq 0)$

1) Se $0 < L < +\infty$

$$f \in L([a,b)) \iff g \in L([a,b))$$

2) Se $L = 0$

$$g \in L([a,b)) \implies f \in L([a,b))$$

$$\left( f \notin L([a,b)) \implies g \notin L([a,b)) \right)$$

3) Se $L = +\infty$

$$f \in L([a,b)) \implies g \in L([a,b))$$

$$\left( g \notin L([a,b)) \implies f \notin L([a,b)) \right)$$

$$\int_1^{+\infty} \frac{x^3}{1+x^4}\, dx \qquad \lim_{x \to +\infty} \frac{\frac{x^3}{1+x^4}}{\frac{1}{x}} =$$

$$= \lim_{x \to +\infty} \frac{x^4}{\cancel{1}+x^4} = 1$$

$$\frac{1}{x} \notin L([1,+\infty)) \implies \frac{x^3}{1+x^4} \notin L([1,+\infty))$$

$$* \int_1^{+\infty} \underbrace{\lg\left(1+\frac{1}{x^p}\right)}_{f_p(x)} dx \qquad p > 0$$

$$\lim_{y\to 0} \frac{\lg(1+y)}{y} = 1 \iff \lim_{x\to+\infty} \frac{\lg\left(1+\frac{1}{x^p}\right)}{\frac{1}{x^p}} = 1$$

teor. confronto asintotico

$$\frac{1}{x^p} \in L([1,+\infty)) \iff f_p \in L([1,+\infty))$$

$$\Updownarrow$$

$$p > 1$$

$*$ è sommabile se (sse) e solo se $p > 1$

iff

$$\lim_{x\to+\infty} \frac{\lg\left(1+\frac{1}{x^p}\right)}{\frac{1}{x^p}} = \lim_{y\to 0^+} \frac{\lg(1+y)}{y}$$

$$y = \frac{1}{x^p}$$

$$\int_1^{+\infty} \frac{1}{[x]^p}\,dx \qquad \text{sommabile} \qquad p>0$$

$$\frac{1}{[x]^p} \overset{?}{\in} L_{loc}([1,+\infty)) \qquad \text{Yes}$$

$\frac{1}{[x]^p}$ è decrescente poiché

$x \longrightarrow [x]$ è crescente

$$\lim_{x\to+\infty} \frac{[x]}{x} = \lim_{x\to+\infty} \frac{x+([x]-x)}{x} \qquad [x] \quad x \quad [x]+1$$

$$= \lim_{x\to+\infty} \frac{\cancel{x}}{\cancel{x}}\left(1+\frac{[x]-x}{x}\right) = 1$$

$$[x] \le x < [x]+1$$

$$0 \le x-[x] < 1 \qquad 0 \ge [x]-x > -1 \quad \cdot \frac{1}{x}$$

$$0 \ge \frac{[x]-x}{x} > -\underset{\downarrow 0}{\frac{1}{x}} \implies \lim_{x\to+\infty} \frac{[x]-x}{x} = 0$$

$$\lim_{x\to+\infty} \frac{\frac{1}{[x]^p}}{\frac{1}{x^p}} = \lim_{x\to+\infty} \frac{x^p}{[x]^p} = \lim_{x\to+\infty}\left(\frac{x}{[x]}\right)^p$$

$$y=\frac{x}{[x]} \qquad = \lim_{y\to 1} y^p = 1$$

$$\frac{1}{[x]^p} \in L([1,+\infty)) \iff \frac{1}{x^p} \in L([1,+\infty))$$

$$\iff p>1$$

$$\lim_{x\to+\infty} \frac{\frac{1}{[x]^p}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{x}{[x]^p} = \lim_{x\to+\infty} \frac{x}{x^p}\,\frac{x^p}{[x]^p}$$

$$= \lim_{x\to+\infty} x^{1-p} = \begin{cases} 0 & \text{se } p>1 \\ 1 & p=1 \\ +\infty & p<1 \end{cases}$$

Per $p \le 1$ $\frac{1}{[x]^p}$ non è integrabile

**Def** Sia $f \in L_{loc}([a,b))$. Allora, se $|f| \in L([a,b))$ diciamo che $f$ è assolutamente integrabile in $[a,b)$.

**Teor** Se $f$ è assolutamente integrabile in $[a,b)$ allora $f$ è integrabile in $[a,b)$.

**Es$_1$** $f(x) = \sin x \dfrac{1}{1+x^2}$ $\qquad [1,+\infty)$

$$|f(x)| = |\sin x| \frac{1}{1+x^2} \le \frac{1}{1+x^2} \in L([1,+\infty))$$

$$\int_1^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to+\infty} \int_1^R \frac{1}{1+x^2}\,dx =$$

$$= \lim_{R\to+\infty} \left(\arctan R - \underbrace{\arctan 1}_{\frac{\pi}{4}}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

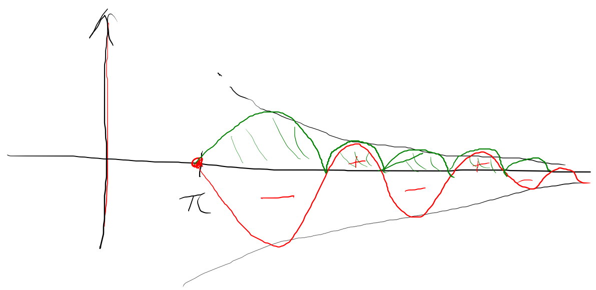
Per il teor del confronto $|f(x)| \in L([1,+\infty))$

$\Rightarrow f \in L([1,+\infty))$

$$f(x) = \frac{\sin x}{x^p} \qquad p > 0$$

$$f \in L[\pi, +\infty) \qquad \not\forall$$

$$|f(x)| \in L[\pi, +\infty) \iff p > 1$$




$$f(x) = \frac{\sin x}{x^p}$$

$$|f(x)| = \frac{|\sin x|}{x^p}$$

$$\int_\pi^x \frac{\sin t}{t^p}\, dt = -\int_\pi^x \frac{\cos' t}{t^p}\, dt =$$

$$= -\frac{\cos t}{t^p}\Big]_\pi^x + \int_\pi^x \cos t \left(\frac{1}{t^p}\right)' dt$$

$$= \frac{1}{\pi^p} - \frac{\cos x}{x^p} - p\int_\pi^x \frac{\cos t}{t^{p+1}}\, dt$$

Per $p > 0$

$$\left|\frac{\cos t}{t^{p+1}}\right| \leq \frac{1}{t^{p+1}} \in L([\pi, +\infty))$$

$$\int_\pi^x \frac{\sin t}{t^p} = \frac{1}{\pi^p} - \underbrace{\frac{\cos x}{x^p}}_{\downarrow\, 0} - p\int_\pi^x \frac{\cos t}{t^{p+1}}\, dt \xrightarrow{x \to +\infty} \ell \in \mathbb{R}$$

$$\ell = \int_\pi^{+\infty} \frac{\sin t}{t^p}\, dt$$