21 dicembre

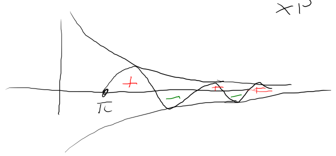
Ieri

$f \in L_{loc}([a,b))$ e $|f| \in L([a,b)) \Rightarrow f \in L([a,b))$

Esempio di $f \in L([\pi,+\infty))$ con $|f| \notin L([\pi,+\infty))$

Abbiamo visto ieri che $f(x) = \frac{\operatorname{sen} x}{x^p} \in L([\pi,+\infty))$

$\forall\, p>0$.



Per $0<p\le 1$ f non è assolutamente integrabile

Supponiamo per assurdo

$$+\infty > \int_\pi^{+\infty} \frac{|\operatorname{sen} x|}{x^p}\,dx = \lim_{R\to+\infty} \int_\pi^R \frac{|\operatorname{sen} x|}{x^p}\,dx$$

$$= \lim_{n\to+\infty} \int_\pi^{(n+1)\pi} \frac{|\operatorname{sen} x|}{x^p}\,dx =$$

$$= \lim_{n\to+\infty} \sum_{j=1}^{n} \int_{j\pi}^{(j+1)\pi} \frac{|\operatorname{sen} x|}{x^p}\,dx \ge$$

$j^p\pi^p \le x^p \le (j+1)^p\pi^p \qquad x^{-p} \ge \frac{1}{\pi^p}\frac{1}{(j+1)^p}$

$$\ge \lim_{n\to+\infty} \sum_{j=1}^{n} \int_{j\pi}^{(j+1)\pi} \frac{|\operatorname{sen} x|}{\pi^p (j+1)^p}$$

$$= \lim_{n\to+\infty} \sum_{j=1}^{n} \frac{1}{\pi^p (j+1)^p} \underbrace{\int_{j\pi}^{(j+1)\pi} |\operatorname{sen} x|\,dx}_{2}$$

$$= \lim_{n\to+\infty} \frac{2}{\pi^p} \sum_{j=1}^{n} \frac{1}{(j+1)^p} \qquad k = j+1$$

$$= \lim_{n\to+\infty} \frac{2}{\pi^p} \sum_{k=2}^{n+1} \frac{1}{k^p}$$

$$= \lim_{n\to+\infty} \frac{2}{\pi^p} \sum_{k=2}^{n+1} \int_k^{k+1} \frac{1}{[x]^p}\,dx$$

($\frac{1}{k^p} = \int_k^{k+1} \frac{1}{[x]^p}\,dx$ perché $[x]=k$ per $k \le x < k+1$)

$$= \lim_{n\to+\infty} \frac{2}{\pi^p} \int_2^{n+2} \frac{1}{[x]^p}\,dx = +\infty$$

$$+\infty > \int_{\pi}^{+\infty} \frac{|\sin x|}{x^p}\,dx \geq \lim_{n\to+\infty} \int_{\pi}^{(n+1)\pi} \frac{|\sin x|}{x^p}\,dx$$

$$\geq \lim_{n\to+\infty} \frac{2}{\pi^p} \int_{2}^{n+2} \frac{1}{[x]^p} = +\infty$$

$+\infty > +\infty$, Assurdo.

$$\int_{n\pi}^{(n+1)\pi} |\sin x|\,dx = 2$$

1) Verifichiamo che $|\sin(x+k\pi)| = |\sin(x)|$ $\forall k \in \mathbb{Z}$

$$|\sin(x+k\pi)| = |\sin(x)\cos(k\pi) + \cos(x)\underbrace{\sin(k\pi)}_{0}|$$

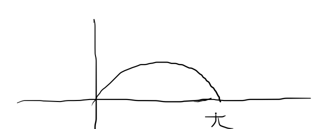
$$= |\sin(x)\,(-1)^k| = |\sin x|$$

2) $$\int_{n\pi}^{(n+1)\pi} |\sin(x)|\,dx =$$

$x = y + n\pi$

$y = x - n\pi$

$dy = dx$



$$= \int_0^{\pi} \overbrace{|\sin(y+\cancel{n\pi})|}^{=|\sin y|}\,dy = \int_0^{\pi} |\sin x|\,dx$$

$$= \int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = \cos(0) - \cos(\pi)$$

$$= 1 - (-1) = 2$$

Teor f integrabile per Darboux in $[a,b] \Rightarrow |f|$ è integrabile per Darboux in $[a,b]$ ed inoltre

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

Esempio $|f|$ integrabile per Darboux $[a,b]$ $\not\Rightarrow$ f integrabile per Darboux in $[a,b]$

Es $[0,1]$

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ -1 & x \notin \mathbb{Q} \end{cases}$$

f Non è integrabile. $|f(x)| \equiv 1$ è integrabile

$$f(x) = 2\left(D(x) - \frac{1}{2}\right) \quad \text{dove } D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

$$\Rightarrow \quad D(x) = \frac{1}{2} f(x) + \frac{1}{2} \qquad \forall x \in [0,1]$$

e se f fosse integrabile per Darboux in $[0,1]$ anche $D(x)$ lo sarebbe

12/9/22 $\qquad a > 0$

1) $$\lim_{x\to+\infty} \frac{\lg(1+x+e^{x^2}) - \int_x^{2x} [t]\,dt}{\int_x^{2x} \operatorname{th}(t^a)\,\frac{1+t^a}{1+t}\,dt} = L_a$$

$$\frac{A\,x^{\cancel{\alpha}}}{B\,x^{\beta}}$$

denom $$\int_x^{2x} \overbrace{\operatorname{th}(t^a)}^{(1+o(t^{-n}))}\,\frac{1+t^a}{1+t}\,dt$$

$\operatorname{th}(t^a) = 1 + o(t^{-n}) \quad \forall n \qquad \operatorname{th}(x) = 1 - 2e^{-2x}(1+o(1))$

$$\text{denom} = \int_x^{2x} \frac{1+t^a}{1+t}\,dt + \int_x^{2x} o(t^{-n})\,\frac{1+t^a}{1+t}\,dt$$

$$1 - \operatorname{th}(x) = 1 - \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\cancel{e^x} + e^{-x} - (\cancel{e^x} - e^{-x})}{e^x + e^{-x}} = 2e^{-x}\,\frac{1}{e^x + e^{-x}}$$

$$= \frac{2e^{-x}}{e^x\,(1+e^{-2x})} = 2e^{-2x}\,\frac{1}{1+e^{-2x}}$$

$$= 2e^{-2x}\,\frac{1}{1+o(1)} = 2e^{-2x}(1+o(1))$$

$$\text{denom} = \int_x^{2x} \frac{1+t^a}{1+t}\,dt + \underbrace{\int_x^{2x} o(t^{-n})\,\frac{1+t^a}{1+t}\,dt}^{= o(x^{-n}) \quad \forall n \in \mathbb{N}} \quad \forall n \in \mathbb{N}$$

$$\frac{1+t^a}{1+t} = \frac{t^a}{t}\,\frac{1+t^{-a}}{1+t^{-1}} = t^{a-1}(1+o(1))$$

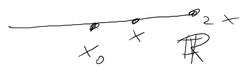
$$\int_x^{2x} o(t^{-n})\,\frac{1+t^a}{1+t}\,dt = \int_x^{2x} o(t^{-m})\,t^{a-1}(1+o(1)) \quad \forall n$$

$$= \underbrace{\int_x^{2x} o(t^{-m})\,dt}_{o(x^{-n+1})} \qquad \forall n$$

$$\lim_{x\to+\infty} \frac{\int_x^{2x} o(t^{-n})\,dt}{x^{-n+1}} = 0$$

$$= \lim_{x\to+\infty} \frac{2\,o(t^{-n})|_{t=2x} - o(t^{-n})|_{t=x}}{(-n+1)\,x^{-n}} = \lim_{x\to+\infty} \frac{o(x^{-n})}{(-n+1)\,x^{-n}} = 0$$

$\cancel{F(x)}$ $\frac{d}{dx}\int_x^{2x} f(t)\,dt =$



$$= \frac{d}{dx}\left[\int_{x_0}^{2x} f(t)\,dt + \int_x^{x_0} f(t)\,dt\right]$$

$$= \frac{d}{dx}\left[\int_{x_0}^{2x} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right] =$$

$$F(x) = \int_{x_0}^{x} f(t)\,dt$$

$$= \frac{d}{dx}\left[F(2x) - F(x)\right] =$$

$$= F'(2x)\,(2x)' - F'(x)$$

$$= 2\,f(2x) - f(x)$$

$$\text{denom} = \int_x^{2x} \frac{1+t^a}{1+t}\,dt + o(x^{-n}) =$$

$$= \int_x^{2x} t^{a-1}\,\frac{1+t^{-a}}{1+t^{-1}}\,dt = \int_x^{2x} t^{a-1}(1+t^{-a})(1-t^{-1}+o(t^{-1}))$$

$$\frac{1}{1+y} = 1 - y + o(y) \qquad y \to 0$$

$$\frac{1}{1+t^{-1}} = 1 - \frac{1}{t} + o\left(\frac{1}{t}\right)$$

$$= \int_x^{2x} t^{a-1}\,(1+o(1))\,dt = \int_x^{2x} t^{a-1} + \int_x^{2x} t^{a-1} o(1)\,dt$$

$$\int_x^{2x} t^{a-1}\,dt = \left.\frac{t^a}{a}\right]_x^{2x} = \frac{2^a x^a}{a} - \frac{x^a}{a} =$$

$$= (2^a - 1)\frac{1}{a}\,x^a$$

$$\int_x^{2x} t^{a-1} o(1)\,dt = \int_x^{2x} o(t^{a-1})\,dt = o(x^a)$$

$$\text{denominator} = (2^a-1)\frac{1}{a}\,x^a + o(x^a) + o(x^{-n}) \qquad \forall n$$

$$= (2^a-1)\frac{1}{a}\,x^a + o(x^a)$$

$$= (2^a-1)\frac{1}{a}\,x^a\left(1 + \frac{o(x^a)}{(2^a-1)\frac{1}{a}x^a}\right)$$

$$= (2^a-1)\frac{1}{a}\,x^a\,(1+o(1))$$

$$\lim_{x\to+\infty} \frac{\text{num}}{(2^a-1)\frac{1}{a}x^a}$$

$$\text{num} = \lg\left(1+x+e^{x^2}\right) - \int_x^{2x} [t] \; dt$$

$$\lg\left(1+x+e^{x^2}\right) = \lg\left(e^{x^2}\left(1+xe^{-x^2}+e^{-x^2}\right)\right)$$

$$= x^2 + \lg\left(1+xe^{-x^2}+e^{-x^2}\right) = x^2 + xe^{-x^2} + e^{-x^2} + o\left(xe^{-x^2}+e^{-x^2}\right)$$

$$= x^2 + o(x^{-n}) \quad \forall n$$

$$\lg(1+y) = y + o(y)$$

$$\frac{xe^{-x^2}}{x^{-10}} = \frac{x^{11}}{e^{x^2}} \xrightarrow[x\to+\infty]{} 0$$

$$\cancel{\int_x^{2x} [t]\, dt = \left.\frac{[t]^2}{2}\right]_x^{2x}}$$

$$\int_x^{2x} [t]\, dt = \int_x^{2x} \left(t + [t] - t\right) dt =$$

$$= \int_x^{2x} t \; dt - \int_x^{2x} \left(t-[t]\right) dt$$

$$= \left.\frac{t^2}{2}\right]_x^{2x} - \int_x^{2x} \left(t-[t]\right) dt$$

$$= \underbrace{2x^2 - \frac{x^2}{2}}_{\frac{3}{2}x^2} - \int_x^{2x} \left(t-[t]\right) dt = \int_x^{2x} [t]\, d$$

$$\text{num} = \lg\left(1+x+e^{x^2}\right) - \int_x^{2x} [t]\; dt =$$

$$= x^2 - \frac{3}{2}x^2 + o(x^{-n}) + \int_x^{2x} \left(t-[t]\right) dt$$

$$\text{numerator} = -\frac{1}{2}x^2 + o(x^2) = -\frac{1}{2}x^2\left(1 + \frac{o(x^2)}{-\frac{1}{2}x^2}\right) \quad \frac{o(x^2)}{-\frac{1}{2}x^2} = o(1)$$

$$= -\frac{1}{2}x^2\left(1+o(1)\right)$$

$$0 \le \int_x^{2x} \underbrace{\left(t-[t]\right)}_{<1} dt < \int_x^{2x} dt = x$$

$$= o(x^2)$$

$$\lim_{x\to+\infty} \frac{\text{num}}{\text{denom}} = \lim_{x\to+\infty} \frac{-\frac{1}{2}x^2 \left(1+o(1)\right)}{\frac{2^a-1}{a}x^a \left(1+o(1)\right)}$$

$$= -\frac{a}{2(2^a-1)} \lim_{x\to+\infty} x^{2-a} = \begin{cases} -\infty & 0<a<2 \\ -\frac{a}{2(2^a-1)} & a=2 \\ 0 & a>2 \end{cases}$$

$$\int_0^1 \operatorname{artg}(x)\, x^2\, dx = \int_0^1 \operatorname{artan}(x) \left(\frac{x^3}{3}\right)' dx =$$

$$= \operatorname{artan} x \,\frac{x^3}{3}\Big]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2}\, dx$$

$$= \frac{\pi}{12} - \frac{1}{3}\int_0^1 \frac{x^3+x}{1+x^2}\, dx + \frac{1}{3}\int_0^1 \frac{x}{1+x^2}\, dx$$

$$= \frac{\pi}{12} - \frac{1}{3}\int_0^1 x\, dx + \frac{1}{6}\int_0^1 \frac{2x}{1+x^2}\, dx$$

$$= \frac{\pi}{12} - \frac{1}{3}\,\frac{1}{2} + \frac{1}{6}\, \lg(1+x^2)\Big]_0^1$$

$$= \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\lg 2$$