$$u^{(n)} + a_{n-1} u^{(n-1)} + \dots + a_0 u = f(t)$$

$$f(t) = \begin{cases} t^m e^{\alpha t} \cos \omega t \\ t^m e^{\alpha t} \sin \omega t \end{cases} \qquad \begin{aligned} & m \in \mathbb{N} = \{0, 1, 2, \dots\} \\ & \alpha \in \mathbb{R}, \quad \omega \in \mathbb{R}_+ = [0, +\infty[\end{aligned}$$

Se $\alpha + i\omega$ è zero di $P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$ di molteplicità s, l'Ansatz da usare è

$$\psi(t) = t^s (A_m t^m + \dots + A_0) e^{\alpha t} \cos \omega t +$$
$$+ t^s (B_m t^m + \dots + B_0) e^{\alpha t} \sin \omega t$$

ESEMPIO 1

$u'' - u = \sin t - t$

$P(\lambda) = \lambda^2 - 1$; gli zeri sono ± 1.

Consideriamo separatamente

$\underset{(a)}{u'' - u = \sin t}$; $\underset{(b)}{u'' - u = t}$

Per (a) Ansatz $\psi_1(t) = A \sin t + B \cos t$

$(m = 0, \ \alpha = 0, \ \omega = 1, \ s = 0)$

$\psi_1' = A\cos t - B\sin t, \quad \psi_1'' = -A\sin t - B\cos t$

$-A\sin t - B\cos t - A\sin t - B\cos t = \sin t$

$-2A\sin t - 2B\cos t = \sin t$

$\Rightarrow B = 0, \quad A = -\frac{1}{2}$

$\Rightarrow \psi_1 = -\frac{1}{2}\sin t$

Per (b) Ansatz $\psi_2(t) = A_1 t + A_0$

$(m = 1, \alpha = 0, \omega = 0, s = 0)$

$\psi_2' = A_1, \quad \psi_2'' = 0$

$0 - A_1 t - A_0 = t \Rightarrow A_0 = 0, \ A_1 = -1$

$\Rightarrow \psi_2 = -t$

Infine $\psi = \psi_1 - \psi_2 = -\frac{1}{2}\sin t + t$

La soluzione generale è

$u(t) = K_1 e^t + K_2 e^{-t} - \frac{1}{2}\sin t + t$

ESEMPIO 2

$$u'' - 4u' + 5u = t e^{2t} \sin t$$

$$P(\lambda) = \lambda^2 - 4\lambda + 5$$

Le radici sono $\lambda_{1,2} = \dfrac{4 \pm i\sqrt{|16-20|}}{2} = \dfrac{4 \pm i\sqrt{4}}{2}$

$$= 2 \pm i$$

La base per l'equazione omogenea è

$$\varphi_1(t) = e^{2t} \cos t \qquad \varphi_2(t) = e^{2t} \sin t$$

Il termine non omogeneo è $f(t) = t e^{2t} \sin t$

$$(m=1, \quad \alpha = 2, \quad \omega = 1, \quad s = 1)$$

L'Ansatz è

$$\psi(t) = t(A_1 t + A_0) e^{2t} \cos t + t(B_1 t + B_0) e^{2t} \sin t$$

Si ha:

$$\psi'(t) = (2A_1 t + A_0) e^{2t} \cos t + 2(A_1 t^2 + A_0 t) e^{2t} \cos t$$
$$- (A_1 t^2 + A_0 t) e^{2t} \sin t$$
$$+ (2B_1 t + B_0) e^{2t} \sin t + 2(B_1 t^2 + B_0 t) e^{2t} \sin t$$
$$+ (B_1 t^2 + B_0 t) e^{2t} \cos t$$

$$\psi' = \left[(2A_1 + B_1)t^2 + (2A_1 + 2A_0 + B_0)t + A_0\right] e^{2t} \cos t$$
$$+ \left[(2B_1 - A_1)t^2 + (2B_1 + 2B_0 - A_0)t + B_0\right] e^{2t} \sin t$$

$$\psi''(t) = \left[2(2A_1 + B_1)t + (2A_1 + 2A_0 + B_0)\right] e^{2t} \cos t$$
$$+ 2\left[(2A_1 + B_1)t^2 + (2A_1 + 2A_0 + B_0)t + A_0\right] e^{2t} \cos t$$
$$- \left[(2A_1 + B_1)t^2 + (2A_1 + 2A_0 + B_0)t + A_0\right] e^{2t} \sin t$$
$$+ \left[2(2B_1 - A_1)t + (2B_1 + 2B_0 - A_0)\right] e^{2t} \sin t$$
$$+ 2\left[(2B_1 - A_1)t^2 + (2B_1 + 2B_0 - A_0)t + B_0\right] e^{2t} \sin t$$
$$+ \left[(2B_1 - A_1)t^2 + (2B_1 + 2B_0 - A_0)t + B_0\right] e^{2t} \cos t$$

$$\psi''(t) = \left[(3A_1 + 4B_1)t^2 + (3A_0 + 8A_1 + 4B_0 + 4B_1)t\right.$$
$$\left. + (4A_0 + 2A_1 + 2B_0)\right] e^{2t} \cos t$$
$$+ \left[(-4A_1 + 3B_1)t^2 + (-4A_0 - 4A_1 + 3B_0 + 8B_1)t\right.$$
$$\left. + (-2A_0 + 4B_0 + 2B_1)\right] e^{2t} \sin t$$

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$$\psi'' - 4\psi' + 5\psi = [4B_1 t + (2A_1 + 2B_0)] e^{2t}\cos t$$
$$+ [-4A_1 t + (-2A_0 + 2B_1)] e^{2t}\sin t$$

$$\begin{cases} 4B_1 = 0 \\ 2A_1 + 2B_0 = 0 \\ -4A_1 = 1 \\ -2A_0 + 2B_1 = 0 \end{cases} \qquad \begin{cases} B_1 = 0 \\ A_1 = -B_0 \\ A_1 = -1/4 \\ A_0 = B_1 \end{cases} \qquad \begin{cases} A_0 = 0 \\ A_1 = -1/4 \\ B_0 = 1/4 \\ B_1 = 0 \end{cases}$$

$$\Rightarrow \quad \psi(t) = -\frac{1}{4} t^2 e^{2t}\cos t + \frac{1}{4} t\, e^{2t}\sin t$$

La soluzione generale è

$$u(t) = K_1 e^{2t}\cos t + K_2 e^{2t}\sin t - \frac{1}{4} t^2 e^{2t}\cos t + \frac{1}{4} t e^{2t}\sin t$$