Lezione 15 dicembre 22

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Visto: Teorema di Dimensione per AL

$f: V \to V'$ AL $\Rightarrow$

$$\dim V = \dim \ker f + \underbrace{\dim \operatorname{Im} f}_{= rg\, f}$$

COROLLARIO: Sia $A \cdot X = 0$ SLO e sia W il ssv di K^n formato delle soluzioni.

Allora $\dim W = n - rg\, A.$

Dim.: $A \cdot X = 0$, $A \in M_{m,n}(K)$

Consideriamo $L_A : K^n \to K^m$

$$L_A\left(\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right) = \underbrace{\underset{m \times n}{A} \cdot \overset{n \times 1}{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}}_{m \times 1} \in K^m$$

Applico Teo di Dim a L_A:

$$\dim K^n = \dim \ker L_A + \underbrace{\dim \operatorname{Im} L_A}_{\boxed{rg\, L_A}}$$

$$n = \dim \ker L_A + \boxed{rg\, A}$$

$$\boxed{n - rg\, A = \dim \ker L_A}$$

$$\ker L_A = \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} : L_A\left(\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right) = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} : A \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} : \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \text{ è soluzione di } A \cdot X = 0 \right\}$$

$$= W$$

$\Rightarrow n - rg\, A = \dim W.$

Altre conseg. del Teor. di Dimensione:

COROLLARIO:

Supp. V, V' SV con $\dim V = \dim V'$.

Allora sono equivalenti le seguenti:

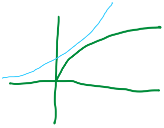
① f è iniettiva, cioè $\ker f = \{0_V\}$

② f è suriettiva, cioè $\operatorname{Im} f = V'$

③ f è isomorfismo (AL biiettiva).

OSS: Questo non vale per funzioni comuni:

$f: \mathbb{R} \to \mathbb{R}$ iniettive e non suriettive



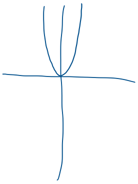
$f(x) = \sqrt{x}$, $D(f) = \mathbb{R}_{\geq 0}$

$f(x) = e^x$, $D(f) = \mathbb{R}$

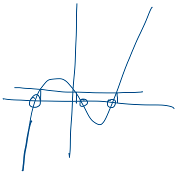
$Im(f) = \mathbb{R}_{>0} \subsetneq \mathbb{R}$

$f: \mathbb{R} \to \mathbb{R}$ suriettive e non iniettive

$f(x) = x^4$



non suriettive



$f(x) = (x-1)(x+1)(x-2)$

DIM.: ① $\Rightarrow$ ②

Supp. f iniettive : $\ker f = \{0_V\}$

$\Rightarrow \dim \ker f = 0$

$\Rightarrow \dim V = \dim \ker f + \dim Im f$

$= 0 + \dim Im f = \dim Im f$

$= \dim V'$

$Im f \subseteq V'$ e hanno la stessa dimensione

$\Rightarrow Im f = V' \Rightarrow$ f suriettive

② $\Rightarrow$ ①

f suriettive $\Rightarrow Im f = V'$

$\Rightarrow \cancel{\dim V} = \dim \ker f + \dim Im f$

$= \dim \ker f + \dim V'$

$= \dim \ker f + \cancel{\dim V}$

$\Rightarrow \dim \ker f = 0 \Rightarrow \ker f = \{0_V\}$

$\Rightarrow$ f iniettive

Infine: ① $\Rightarrow$ ② $\Rightarrow$ ③ per def. di biiettività

② $\to$ ① $\Rightarrow$ ③

MATRICI ASSOCIATE A AL

Def: Sia $f: V \longrightarrow V'$ AL

$\dim V = n$, $\dim V' = m$

Siano $\mathcal{B} = \{v_1, \dots, v_n\}$ base di V

$\mathcal{C} = \{w_1, \dots, w_m\}$ base di V'

La MATRICE ASS. a f nelle basi $\mathcal{B}$ e $\mathcal{C}$ è

$$\underset{m \times n}{M_{\mathcal{C}}^{\mathcal{B}}(f)} := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

le cui COLONNE sono le COORDINATE di $f(v_j)$ rispetto alla base $\mathcal{C}$, $\forall\, v_j \in \mathcal{B}$

coordinate di $f(v_1)$ nella $\mathcal{C}$

coordinate di $f(v_n)$ nella base $\mathcal{C}$

Esempio: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right) = \begin{pmatrix} c_1 + c_2 \\ c_1 - 3c_2 \end{pmatrix}$$ è AL

Fisso base di $\mathbb{R}^2$: $\mathcal{E}$ canonica

$\mathbb{R}^2$ come codominio: $\mathcal{E}$ canonica

$$\underset{2\times 2}{M^{\mathcal{E}}_{\mathcal{E}}(f)} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \qquad =$$

$\downarrow$

$$f(e_1) = f\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}$$

$$f(e_2) = f\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\-3\end{pmatrix}$$

OSS: $$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - 3c_2 \end{pmatrix}$$

PROP: $\underset{\underset{\text{base}}{\mathcal{B}}}{f: V} \to \underset{\underset{\text{base}}{\mathcal{C}}}{V'}$ AL, cons. $M^{\mathcal{B}}_{\mathcal{C}}(f)$

Allora se $v \in V$ ha coordinate $\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$ nella $\mathcal{B}$

cioè $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$, allora $f(v)$

ha coordinate $\begin{pmatrix}\beta_1\\ \vdots \\ \beta_m\end{pmatrix} = M^{\mathcal{B}}_{\mathcal{C}}(f) \cdot \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$

$$f(v) = \beta_1 w_1 + \cdots + \beta_m w_m$$

Dim. $f(v) = f(\alpha_1 v_1 + \cdots + \alpha_n v_n)$

(AL1) $|$

$= f(\alpha_1 v_1) + \cdots + f(\alpha_n v_n)$

(AL2) $|$

$= \alpha_1 f(v_1) + \cdots + \alpha_n f(v_n)$

$$= \alpha_1 (a_{11} w_1 + a_{21} w_2 + \dots + a_{m1} w_m) +$$

entrate della prima colonna

$$+ \alpha_2 (a_{12} w_1 + a_{22} w_2 + \dots + a_{m2} w_m) +$$

$$\vdots$$

$$+ \alpha_n (a_{1n} w_1 + a_{2n} w_2 + \dots + a_{mn} w_m)$$

$\mathcal{C} = \{w_1, \dots, w_m\}$ base di V'

$$= \underbrace{(\alpha_1 a_{11} + \alpha_2 a_{12} + \dots + \alpha_n a_{1n})}_{= \beta_1} w_1 +$$

$$+ \underbrace{(\alpha_1 a_{21} + \alpha_2 a_{22} + \dots + \alpha_n a_{2n})}_{\beta_2} w_2 +$$

$$\dots$$

$$+ \underbrace{(\alpha_1 a_{m1} + \alpha_2 a_{m2} + \dots + \alpha_n a_{mn})}_{\beta_m} w_m$$

$$\underbrace{(a_{11} \quad a_{12} \quad \dots \quad a_{1n})}_{= A_{(1)}} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\boxed{A = M^{\mathcal{B}}_{\mathcal{C}}(f)}$$

$$A_{(2)} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$A_{(m)} \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \\ \beta_m \end{pmatrix} = A \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$$

ATTENZIONE: $M^{\mathcal{B}}_{\mathcal{C}}(f)$ DIPENDE

DALLA SCELTA delle basi $\mathcal{B}$ e $\mathcal{C}$

Esempio: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f\left(\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right) = \begin{pmatrix} c_1 + c_2 \\ c_1 - 3c_2 \end{pmatrix}$$

base di $\mathbb{R}^2$: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

$$\mathcal{C} = \left\{ \overbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}^{w_1}, \overbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}^{w_2} \right\}$$

$$M^{\mathcal{B}}_{\mathcal{C}}(f) = \begin{pmatrix} -4 & -12 \\ -2 & -7 \end{pmatrix}$$

$f\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ e cerco le coordinate nelle $\mathcal{C}$

$$\begin{pmatrix} 2 \\ -2 \end{pmatrix} = a_{11} \cdot w_1 + a_{21} \cdot w_2 = a_{11}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + a_{21}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -a_{11} + a_{21} \\ a_{21} \end{pmatrix}$$

$$\Rightarrow \begin{cases} -a_{11} + a_{21} = 2 \\ a_{21} = \boxed{-2} \end{cases} \Rightarrow a_{11} = a_{21} - 2 = \boxed{-4}$$

$$f\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ -7 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -7 \end{pmatrix} = a_{12}\, w_1 + a_{22}\, w_2 = a_{12}\begin{pmatrix} -1 \\ 0 \end{pmatrix} + a_{22}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -a_{12} + a_{22} \\ a_{22} \end{pmatrix}$$

$$\begin{cases} -a_{12} + a_{22} = 5 \\ a_{22} = \boxed{-7} \end{cases}, \quad a_{12} = a_{22} - 5 = \boxed{-12}$$

Oss: Cons. l'AL nulla:

$f: V \to V'$

$f(v) = 0_{V'}$ vettore nullo di V'

$\forall v \in V$

Fisso $\mathcal{B}$ in V, $\mathcal{C}$ in V'

$$M^{\mathcal{B}}_{\mathcal{C}}(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

è la MATRICE NULLA in TUTTE LE BASI

$\downarrow$

coordinate di $f(v_1)$ nella $\mathcal{C}$
$= 0_{V'}$

Es: Sia $f = Id_V : \underset{\mathcal{B}}{V} \longrightarrow \underset{\mathcal{B}}{V}$

$f(v) = v$

Fisso $\mathcal{B}$ di V, $\mathcal{B} = \{v_1, \dots, v_n\}$

$$M^{\mathcal{B}}_{\mathcal{B}}(Id_V) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = I_n$$

$f(v_1) = v_1 = 1 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$

$f(v_2) = v_2$

ALTRO USO DELLE $M^{\mathcal{B}}_{\mathcal{C}}(f)$:

Prop. $f : \underset{\mathcal{B}}{V} \to \underset{\mathcal{C}}{V'}$ AL, $M^{\mathcal{B}}_{\mathcal{C}}(f)$

$$\ker f = \{ v = \alpha_1 v_1 + \dots + \alpha_n v_n \mid f(v) = 0_{V'} \} \quad (\ast)$$

cioè

$$\left. M^{\mathcal{B}}_{\mathcal{C}}(f) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$= \left\{ v \;\middle|\; \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ è soluz. di } \underbrace{M^{\mathcal{B}}_{\mathcal{C}}(f) \cdot X = 0}_{SLO} \right\}$$

$$\mathrm{Im} f = \left\{ w = \beta_1 w_1 + \dots + \beta_m w_m \;\middle|\; \exists\, v \in V \text{ t.c. } w = f(v) \right\}$$

$$\exists \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \text{ t.c. } \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = M^{B}_{e}(f) \cdot \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \}$$

$$= \left\{ \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} : M^{B}_{e}(f) \cdot X = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} \text{ è COMPATIBILE} \right\}$$

CONS.: Da ⊛, abbiamo

$$\dim \ker f = \dim W$$

W soluzioni di $M^{B}_{e}(f) \cdot X = 0$

$$=\left(\begin{array}{l} \dim W = n - \mathrm{rg}\, M^{B}_{e}(f) \\ \dim \ker f = n - \mathrm{rg} f \end{array} \right) = \Rightarrow$$

$$\mathrm{rg} f = \mathrm{rg}\, M^{B}_{e}(f)$$ PER QUALUNQUE SCELTA DELLE BASI

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Esempio: $f: \mathbb{R}^2 \to \mathbb{R}^2$

$$f\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}, \quad f\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}1\\1\end{pmatrix}$$

$$f\left(\begin{pmatrix}c_1\\c_2\end{pmatrix}\right) = \begin{pmatrix}c_1+c_2\\c_1+c_2\end{pmatrix} = (c_1+c_2) \cdot \begin{pmatrix}1\\1\end{pmatrix}$$

$$\mathrm{Im} f = \mathrm{Span}\left(\begin{pmatrix}1\\1\end{pmatrix}\right), \quad \dim \mathrm{Im} f = 1 = \mathrm{rg} f$$

$$\mathcal{E} = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

$$M^{\mathcal{E}}_{\mathcal{E}}(f) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = A$$

$$\mathrm{rg} A = 1$$

ker f : soluzioni $A \cdot X = 0$

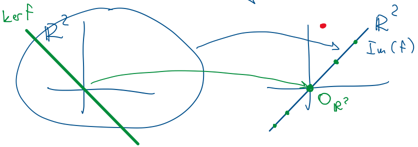
$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases}$$

gen solz. $x_2 = t$, $x_1 = -t$: $\begin{pmatrix}-t\\ t\end{pmatrix}$

$$\ker = \left\{ \begin{pmatrix}-t\\ t\end{pmatrix} : t \in \mathbb{R} \right\} = \text{Span}\left(\begin{pmatrix}-1\\ 1\end{pmatrix}\right) \neq \{0\}$$

$$\dim \ker f = 1 = n - \text{rg} A = 2 - 1$$



$$\text{Im} f = \text{Span}\left(\begin{pmatrix}1\\ 1\end{pmatrix}\right) \not\ni \begin{pmatrix}0\\ 1\end{pmatrix}$$

infatti: $A \cdot X = \begin{pmatrix}0\\ 1\end{pmatrix}$ non è compatibile

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \cdot \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$$

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{cases}$$